# CONFORMALLY InVARIANT PROBABILITY FOR THE NON-MATHEMATICIAN 

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## 1. INTRODUCTION

The goal of this essay is to explain in non-technical terms why the study of "conformally invariant two-dimensional statistical physics models" is compelling and beautiful. One simple reason is that it provides the opportunity to work with aesthetically appealing pictures, such as the one shown in Figure 1.1.


Figure 1.1: An "SLE sunrise" (from Jason Miller and Scott Sheffield's Imaginary Geometry IV). This picture was made in the course of mathematical inquiry, but the strokes are reminiscent of impressionist art!

More importantly, however, the objects of interest are also mathematically beautiful. There has been much progress over the last 15 years in discovering a variety of relationships between apparently unrelated ideas by demonstrating connections to a universal object ${ }^{1}$-a mathematical construct which captures the behavior of a wide class of models. This universal object is a random path called the Schramm-Loewner evolution (SLE), introduced by Oded Schramm ${ }^{2}$ in 1999. In Section 2 , we will discuss the idea of universality in

[^0]a more familiar context. In Section 3, we will describe two important and simple models which are related to SLE. In Section 4, we will discuss SLE and its relationship to these models.

## 2. Universality

You might recall that many real-world phenomena are well modeled by a bell curve. For example, suppose we flip 100 coins and record the number of heads; let's say we get 53 heads. Maybe we do the experiment again and this time our outcome is 42 heads. If we repeat this experiment many times and keep count of how many times we obtain each outcome, we can represent the results in a bar chart as shown in Figure 2.1(a). A remarkable property of the resulting bar chart is that it will almost certainly end up looking very similar to the bell curve shown in red. The more we repeat the experiment, the more closely the resulting bar chart will resemble a bell curve.

Suppose instead that we roll 100 dice and record the total number of dots showing on the tops of all the dice (in other words, we add up the 100 individual die rolls). Repeating this experiment many times also gives a histogram shaped like a bell curve (Figure 2.1(b)). In fact, the bell curve shows up throughout the natural and social sciences when experimenters measure things like heights of animals or test performance of subjects in a study (see the Wikipedia article on the normal distribution ${ }^{(3)}$.

(a) Number of heads in 100 coin flips

(b) Total for 100 die rolls

Figure 2.1: Suppose we repeat the following experiment 10,000 times: flip 100 coins and record the number of heads. We keep a tally of the outcomes of these 10,000 experiments and make a bar chart as shown in panel (a). For example, the bar in the middle indicates that we got exactly 50 heads about 800 times. The bar above " 60 " shows that we only got 60 heads about 100 times. Observe that the tops of the histogram bars are very close to the bell curve shown in red. Panel (b) shows the resulting bar chart for 10,000 trials of the following experiment: we roll 100 dice, record the total number of dots showing. We get the same bell curve shape but centered at a different place on the number line and having a different height and width.

The central limit theorem explains why the bell curve shows up in so many different contexts. It gives precise mathematical meaning to the idea that you always get a bell

[^1]curve as long as there is enough independence in the experiment. For example, we get a bell curve in the coin flipping experiment because the coin flips are independent, which means that the outcome for a particular coin has no effect on the other coins. We could say that some of the details (such as whether we're using coins or dice) are "washed out" in the end-we always get the same bell curve shape.

Observe that the bell curve describes experiments with numerical outcomes. In other words, a dice total or head count is represented by a number. It is also possible to conduct an experiment where the outcome is a path instead of a number (see Section 3). We will see that SLE plays a similar role for certain types of path-valued experiments as the bell curve does for numerical experiments.

## 3. Discrete models

We describe two ways to obtain a random path: the loop-erased random walk, and the percolation interface. We call them discrete models because the paths live on a grid or honeycomb lattice with finitely many points.
3.1 Loop-erased random walk. Consider a bug starting in the middle of a square grid and randomly walking around until it reaches one of the points on the edge of the grid (see Figure 3.1). We assume that at each step, each of the four nearest neighbors is equally likely to be visited next, and each step is made independently of the others. In other words, the bug makes its successive steps by repeatedly rolling a four-sided die with the possible outcomes "up", "down", "left", or "right". The bug might revisit the same location, but whenever it does it erases the loop made since the previous visit to that point. The resulting random path is called a loop-erased random walk. An example on a large grid is shown in Figure 3.3(a).


Figure 3.1: To make a loop-erased random walk, we start a bug at the large point in the middle and ask it to take random steps up, down, left, or right. Each step is equally likely, and all the choices are made independently. In the path shown in the figure, the bug revisits a point after five steps, and upon doing so it erases the loop it took to get there (shown in red). It then steps left, down, and back up, at which point it erases the two-step "loop" it has just made. From this point, the bug happens to make its way to the edge without forming another loop. The loop-erased random walk is the portion of the path which was not erased (shown in black).
3.2 Percolation interface. For a second example of a random path, consider starting with a parallelogram-shaped block of a honeycomb lattice (see Figure 3.2). Color each hexagon on the top and left edges black, and color each hexagon on the bottom and right edges red. Now, for each other hexagon in the lattice, flip a coin and color the hexagon black if the coin comes up heads and red if it comes up tails. This coin-flipping way of coloring hexagons is called percolation, and it is a simple example of a statistical physics model. Now imagine starting in the bottom-left corner and drawing the path that at every step has a red hexagon to the right and a black hexagon to the left. In other words, whenever the hexagon directly in front of us is red, we turn left. Whenever the hexagon directly in front of us is black, we turn right. In Figure 3.2, for example, the turns are "right", "left", "left", "left", "right", and so on. We can see that we can always continue drawing such a path until it exits the parallelogram at the top-right corner (see the figure caption for more discussion of this point). This random path across the parallelogram is called a percolation interface. An example of a percolation interface on a large grid is shown in Figure 3.3(b).


Figure 3.2: There is a unique path (shown in yellow), called the percolation interface, which crosses the parallelogram and separates red hexagons from black hexagons. This path is obtained by starting in the lower left corner between a black and a red hexagon. We turn left whenever we see a red hexagon straight ahead, and we turn right whenever we see a black hexagon straight ahead. For example, in panel (b) the path turns left for its third step because the hexagon straight ahead, indicated with a question mark, is red. This way the path always has a black hexagon to its left and a red hexagon to its right. Observe that the path can't exit on the top or left sides of the parallelogram, because that would require the path to go between two black hexagons. Similarly, the path can't exit on the right or bottom because it would have to go between two red hexagons. Therefore, the path can only exit at the top-right corner.
3.3 Comparing the two models. Observe that the manner in which the loop-erased random walk and the percolation interface are formed is totally different. These models have very little in common, and there seems to be no reason for us to expect that they have anything to do with one another.

Furthermore, if we take a look at Figure 3.3, we see that the loop-erased random walk and the percolation interface seem to behave very differently. The loop-erased random


Figure 3.3: (a) A loop-erased random walk on a $240 \times 240$ grid, and (b) a percolation interface on a $160 \times 160$ honeycomb lattice.
walk is pretty wiggly, but it doesn't seem to be hitting itself and it definitely isn't coming near the boundary until the very end. By contrast, the percolation interface seems to be (almost) hitting itself and the boundary a lot, and it is very wiggly. Any unifying framework for these two random paths would have to be flexible enough to accommodate these very different types of behavior.

There are two features that the loop-erased random walk and the percolation interface do have in common. First, each is a path that by its nature cannot cross itself. Second, each arises from a model possessing a lot of independence (very loosely speaking): for the loop-erased random walk, the walker's steps are independent; for percolation, the colors of the hexagons are independent. These very general properties turn out to be enough to describe the paths using SLE.

## 4. SLE

SLE is a random path that grows with time (see Figure 4.1). For this reason, it's natural to use an animation to illustrate the basic idea of how an SLE path is made. If you're viewing this PDF in Adobe Reader, then you can see the animation in Figure 4.3 below. Otherwise, you can find the animation at

> http://math.mit.edu/~sswatson/sleanimation.html.

The construction of SLE is as follows. Draw a horizontal line in the plane, pick a point on the line. Imagine a person walking around randomly on the line, starting at the point you picked. This walker takes extremely tiny random steps left or right, but steps very frequently, so that the total distance moved every unit of time isn't tiny. The location after one time unit is a random quantity described by a bell curve (see Section 2). Since steps left and right are equally likely, the bell curve is centered at the starting point. The width of the bell curve is determined by how frequently the walker is stepping, and it will be

(a) $k=8 / 3$

(b) $\kappa=6$

Figure 4.1: Simulations of Schramm-Loewner evolution paths by Tom Kennedy, for $\kappa=8 / 3$ and $\kappa=6$, from Numerical Computations for the Schramm-Loewner Evolution. SLE is an example of a fractal, which means that the path is jagged in a certain mathematical sense.
convenient to allow this "speed" to vary. The speed is described in terms of a positive number which is always denoted by $\kappa$ (this is the Greek letter kappa). The larger $\kappa$ is, the faster the walker is stepping. The mathematical term for $\kappa$ is diffusivity, and the term for this kind of walk is Brownian motion.

Now we will further imagine that all the points above the line are watching the walker's movements and responding with movements of their own. In particular, each point moves as if it were subjected to a force with direction shown in Figure 4.2, depending on the walker's location. Points directly above the walker are pulled straight down, while those to above and to the left are pulled down and to the left, and those above and to the right are pulled down and to the right. It's a bit like gravity is pulling the points toward the walker, except that in the horizontal direction the force is pushing the points away from the walker. It can happen that a point moving according to these rules collides with the walker. Alternatively, it is possible that the point will end up getting pushed to the side and never collide with the walker. For any time $t$ we can ask: "show me the original locations of all the points that have collided with the walker by time $t$. ." The answer to this question is a set of points which grows in time, and it turns out to give us a path which we define to be SLE. We sometimes use a subscript and write SLE $_{k}$ to remind ourselves that SLE depends on the walker's speed $\kappa$.


Figure 4.2: The points above the line are subjected to a force with direction indicated by the arrows. The point marked $W_{t}$ is the location of the walker at time $t$. Note that the forces being applied to the points depend on the location of the walker and move along with the walker.


Figure 4.3: A picture/animation illustrating the construction of SLE. (Click the picture to make the animation play. If it doesn't work try this link.) The random walker is the green dot in the right picture, and as it moves, the points in the grid above move as well. The points to the left and right get pushed farther away, but the points directly above the walker get pulled straight down. As the points are moving, sometimes a collision occurs between the green dot and one of the grid dots. At that moment in time, the point's original location is highlighted in the left picture and added to the SLE path, shown in red.
In the actual construction of SLE, we don't use a grid: all the points above the line are moving, and at every instant in time some point is colliding with the walker and having its original location added to the SLE path. We also note that, while the grid is useful for explaining the idea of the construction, but it doesn't do a good job of producing paths that actually look like SLE. For more accurate pictures see Figure 4.1.

It turns out, perhaps surprisingly, that the behavior of the $\mathrm{SLE}_{\kappa}$ path depends drastically on the value of $\kappa$. As summarized in Figure 4.4, the properties of the path are quite different when (i) $\kappa$ is between 0 and 4 , (ii) $\kappa$ is between 4 and 8 , and (iii) $\kappa$ is above 8 . In case (i) the path is simple, which means that it doesn't hit itself. In case (ii), the path is not simple, and in case (iii) the path fills up all the space above the line!


$\mathrm{SLE}_{K}$ self-intersects but isn't space-filling

$\mathrm{SLE}_{\kappa}$ is space-filling

Figure 4.4: Depending on the value of $\kappa$, the path has different properties. When $\kappa=0$, the random walker doesn't move at all, and the SLE grows straight up. When $0<\kappa \leq 4$, the SLE path doesn't intersect itself. When $4<\kappa<8$, the path intersects itself (but doesn't cross; it just touches). When it does so, it pinches off whole regions, depicted here in red. When $\kappa \geq 8$, the SLE path passes through every point above the line! As an example, the SLE sunrise back in Figure 1.1 depicts an SLE $_{128}$ by affixing orange and yellow "markers" to the two sides of the path as it maneuvers around in the square.

Based on these properties, we would guess that if the loop-erased random walk path is described by an SLE $_{k}$, it would be for a value of $\kappa$ between 0 and 4 . This is because the path in our simulation doesn't seem to hit itself. If the percolation interface is described by an SLE $_{\kappa}$, it will probably be for a value of $\kappa$ between 4 and 8 , because the path seems to hit itself a lot. Both of these guesses turn out to be correct, and in particular the correct values of $\kappa$ are 2 and 6 , respectively. As we consider a loop-erased random walk on an increasingly fine grid (where the squares are really small), the behavior of the walk is described more and more accurately by an SLE $_{2}$ path. Similarly, as we consider percolation on an increasingly fine grid, the behavior of the interface is described more and more accurately by an SLE $_{6}$ path. These results were proved by Gregory Lawler, Oded Schramm, and Wendelin Werner ${ }^{4}$ (loop-erased random walk, $\kappa=2$ ), and by Stanislav Smirnov ${ }^{4}$ (percolation interface, $\kappa=6$ ).

In this way, we see that the loop-erased random walk and the percolation interface can be described in the same framework by simply adjusting the speed of the SLE random walker!

## 5. CONFORMAL INVARIANCE

We have seen three different kinds of domains that a path might live in: a square, a parallelogram, and a half-plane (all the points above a line). How can we compare paths in these different domains? A satisfactory answer to this question is crucial for understanding what is meant by the idea that SLE $_{2}$ describes the loop-erased random walk and SLE 6 describes the percolation interface.


Figure 5.1: The half annulus domain on the left can be mapped one-to-one onto the triangular domain in an angle-preserving way. This means that every point in the domain corresponds to exactly one point in the triangular domain and vice versa. It also means that any curves meeting at right angles, like the ones shown in red and blue, map to curves that also meet at right angles. The map shown here sends three of the four "corners" in the half-annulus to the three corners of the right triangle. The correspondence represented in this figure is an example of a conformal map.

One possible answer uses a tool from the field of complex analysis called conformal mapping 5 Imagine that we have two cutouts from a rubber sheet (with neither having holes in

[^2]the middle, like the two regions shown in Figure 5.1), and imagine that the first one has a picture on it. The Riemann mapping theorem tells us that it is always possible to stretch the first cutout to match the shape of the second cutout in such a way that the picture doesn't look distorted if we zoom in and look at a small patch ${ }^{6}$. A bit more precisely, the theorem says that any domain without holes can be mapped one-to-one onto any other domain without holes in a smooth way that preserves angles inside the domain (see Figure 5.1). This gives us one way of comparing paths in different domains: we create our path in one domain, and we apply a conformal map from the first domain to the second. The result is a new path in the second domain (see Figure 5.2). This idea turns out to be very useful, because models that are well-approximated by SLE have the following conformal invariance property: running the model in a domain $D_{2}$ is essentially the same as running the model in a domain $D_{1}$ and applying a conformal map from $D_{1}$ to $D_{2}$.


Figure 5.2: For any path in the half annulus domain on the left (such as the smooth one shown in blue), we obtain a path in the triangular domain by applying the conformal map described in Figure 5.1 to all the points on the path.

With the idea of conformal invariance in mind, we can make the statement about the relationship between $\mathrm{SLE}_{6}$ and the percolation interface more precise. We are saying that if we consider an $\mathrm{SLE}_{6}$ path in a half-plane and conformally map it to the parallelogram, then we obtain a random path which is almost the same as a percolation interface with lots of tiny hexagons. For example, this means that if we ask a question like "what is the probability of the path passing within a distance of 0.01 of the point $(0,0.4)$ ?", then the answers for the SLE $_{6}$ path conformally mapped to the parallelogram and for the percolation interface in the parallelogram are very close. If we make the hexagons smaller and more numerous, the answers will get even closer.

## 6. CONCLUSION

We have seen that there are natural discrete models which are closely related to SLE $_{2}$ and $\mathrm{SLE}_{6}$. Other well studied models are related to $\mathrm{SLE}_{8 / 3}, \mathrm{SLE}_{3}, \mathrm{SLE}_{4}, \mathrm{SLE}_{16 / 3}$, and $\mathrm{SLE}_{8}$. Such a physical map represents an angle-preserving correspondence (in other words, a conformal map) between points on the earth's surface and points on the page.
${ }^{6}$ Of course, the whole picture will be distorted. Only by looking at a very small piece does the stretched picture appear the same as the original. For example, the overall shape of a flat world map and a globe are very different, but local geographies appear the same.


Figure 6.1: A random surface called the Gaussian free field. One can find all sorts of SLE paths by exploring the contours of this surface according to various rules.

SLE shows up so ubiquitously that one learns to look for connections to SLE whenever considering models that one suspects might be conformally invariant.

Many calculations are easier to do for SLE, so the correspondence with discrete models provides a powerful computational tool for analyzing them. For example, consider two hexagons in a percolation honeycomb lattice which are very far apart. What is the probability that there is a path that would enable you to walk from one hexagon to the other using only red hexagons along the way? Using the relationship between the percolation interface and SLE $_{6}$, Lawler, Schramm, and Werner were able to show that this probability goes to 0 like $\frac{1}{R^{5 / 48}}$, where $R$ denotes the distance between the two hexagons. Many other questions can be answered similarly, by translating a question about a discrete model into a question about SLE, which can in turn be translated into a question about the random walker generating the SLE. Since random walks (and, more precisely, Brownian motion) are well understood, this reduction is often helpful.

Many open problems remain in the study of SLE and conformally invariant discrete models. For example, many more discrete models are conjectured to converge to SLE than have actually been proved. The problem is that each proof requires its own insight particular to the model and even particular to the lattice (square versus honeycomb, say). In the context of averaging random numbers, the central limit theorem explains why universal behavior arises and provides a broadly applicable tool for proving that data are well represented by a bell curve. By contrast, our understanding of SLE demonstrates universality in certain cases where we can prove it by hand, but comes short of providing an all-purpose tool or a satisfactory explanation of why we observe conformal invariance in such a wide variety of models. A mathematical development in this direction would be very exciting indeed.

Apart from the relationship to discrete models discussed in this essay, SLE can also be viewed as a contour of a random surface called the Gaussian free field (see Figure 6.1), which plays an important role in theoretical physics. If we think of the random surface as a mountain range, the contours are the paths followed by a mountain climber seeking to stay at the same altitude. These are also the paths drawn on a topographical map of the mountain range. If we instruct the mountain climber to follow different rules which per-
mit changing altitudes, we can obtain all sorts of other types of SLE paths called Gaussian free field flow lines. Many tools have been developed for understanding how these flow lines interact, and they can be used to better understand properties of SLE. This abundance of new ideas gives rise to new questions and provides tools for solving old ones. See Imaginary Geometry I-IV by Jason Miller and Scott Sheffield for many more striking pictures like Figure 1.1 and a thorough development of this perspective.

For further reading about conformally invariant models, Vincent Beffara's lecture notes discuss more examples and are much more mathematically detailed:
http://perso.ens-lyon.fr/vincent.beffara/assets/Buzios.pdf

Wendelin Werner's lecture notes from the 2002 Saint-Flour summer school also cover the topics we have discussed in more detail:
http://arxiv.org/pdf/math/0303354v1.pdf

For more details about the properties of SLE and its relationship to other continuous random objects, see Gregory Lawler's book Conformally invariant processes in the plane.:
http://www.ams.org/publications/authors/books/postpub/surv-114
All of these works include further references to the literature.
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[^0]:    ${ }^{1}$ Italics should be taken to indicate that a term is new and that an explanation of the term is forthcoming.
    ${ }^{2} \mathrm{He}$ called the process the Stochastic Löwner Evolution.

[^1]:    ${ }^{3}$ References are hyperlinked.

[^2]:    ${ }^{4}$ Wendelin Werner and Stanislav Smirnov received Fields Medals in 2006 and 2010, respectively, for this and related work. The Fields Medal is the premier prize in mathematics.
    ${ }^{5}$ The idea of conformal mapping also arises in cartography. It is desirable to represent a portion of the

