18.022 Recitation Handout (with solutions)
10 December 2014

Solution. *For solutions, see page 486 of https://math.berkeley.edu/~strain/170.S13/cov.pdf* □

1. Consider a rotating cylindrical container of radius R with a vertical axis, filled with liquid mercury up to height h. In this problem, we'll use calculus of variations to find the shape of the surface of the liquid.

(a) Assume that the surface of the liquid is described by the equation $z = f(r)$ in cylindrical coordinates, where f is some function. Sketch a graph of f based on physical intuition. Indicate the locations of R and h on your axes.

(b) Calculate the potential energy V of the liquid, assuming the density of the liquid is σ and using g to denote the gravitational constant. Express your answer as an integral in involving f.

(c) Calculate the kinetic energy T of the liquid, assuming the angular velocity of the cylinder is ω. Express your answer as an integral in involving f.

(d) Calculate the volume of the liquid, as an integral in involving f.

(e) Hamilton's principle says that the shape f will minimize $T - V$. Also, a generalization of Lagrange multipliers along with the Euler-Lagrange equations implies the following. The extrema $f(r)$ of $\int_{r_1}^{r_2} G(r, f, f')\,dr$ subject to the constraint $\int_{r_1}^{r_2} H(r, f, f')\,dr = C$ for some constant C are solutions of the system

$$\frac{d}{dr}\left(\frac{\partial(G - \lambda H)}{\partial f'}\right) = \frac{\partial(G - \lambda H)}{\partial f}$$
$$\int_{r_1}^{r_2} H(r, f, f')\,dr = C.$$

Note that f' is regarded as an independent variable. Put together the answers from the previous three parts to find f.

(f) Explain why your answer shows that the shape of the surface is independent of the density of the liquid.