18.022 Recitation Handout (with solutions)
8 December 2014

Let $D \subset \mathbb{R}^2$ be the region enclosed by the curve $r = g(\theta)$, for some C^1, non-negative $g\colon \mathbb{R} \to \mathbb{R}$ such that $g(x + 2\pi) = g(x)$ for all $x \in \mathbb{R}$.

1. Calculate the length of ∂D, the boundary of D. Express your answer as an integral involving g and its first derivative.

Solution. We parametrize the curve by $\mathbf{r}(\theta) = g(\theta)(\cos\theta, \sin\theta)$ for $\theta \in [0, 2\pi]$. Then

$$\mathbf{r}'(\theta) = \frac{\partial g}{\partial \theta}(\theta)(\cos\theta, \sin\theta) + g(\theta)(-\sin\theta, \cos\theta)$$

and

$$|\mathbf{r}'(\theta)|^2 = \left(\frac{\partial g}{\partial \theta}(\theta)\right)^2 + g(\theta)^2.$$

Hence

$$\text{length}(\partial D) = \int_0^{2\pi} \sqrt{\left(\frac{\partial g}{\partial \theta}(\theta)\right)^2 + g(\theta)^2}\,\mathrm{d}\theta.$$

□

2. Let

$$(x(\theta), y(\theta)) = (g(\theta)\cos\theta, g(\theta)\sin\theta)$$

be a parametrization of ∂D. Calculate the length of ∂D again, but this time express the answer as an integral involving the derivatives of x and y.

Solution. The velocity is now $(x'(\theta), y'(\theta))$, and so

$$\text{length}(\partial D) = \int_0^{2\pi} \sqrt{x'(\theta)^2 + y'(\theta)^2}\,\mathrm{d}\theta.$$

□

3. Calculate the length of ∂D for the case that $g(\theta) = 1 - \cos\theta$.

Solution. Note that $g'(\theta) = \sin\theta$. Hence

$$\begin{aligned}
\ell(a,b) &= \int_0^{2\pi} \sqrt{\left(\frac{\partial g}{\partial \theta}(\theta)\right)^2 + g(\theta)^2}\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \sqrt{\sin^2\theta + 1 - 2\cos\theta + \cos^2\theta}\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \sqrt{2(1-\cos\theta)}\,\mathrm{d}\theta \\
&= 8.
\end{aligned}$$

□

4. In the remainder of this problem we will prove an important theorem, called the *isoperimetric inequality* (this proof is due to E. Schmidt, from 1938): it states that the length of the boundary of any shape on the plane is at least equal to the square root of 4π times its area.

Let C be the unit circle. Explain why

$$\text{area}(D) + \pi = \oint_{\partial D} (0, x) \cdot \mathrm{d}\mathbf{s} + \oint_C (-y, 0) \cdot \mathrm{d}\mathbf{s}. \tag{1}$$

Solution. This follows from Green's Theorem. □

5. Assume henceforth that $g(x) \leq 1$ and $g(0) = g(\pi) = 1$. Show that

$$(x(\theta), w(\theta)) = \begin{cases} (g(\theta)\cos\theta, \sqrt{1 - g(\theta)^2\cos^2\theta} & 0 \leq \theta \leq \pi \\ (g(\theta)\cos\theta, -\sqrt{1 - g(\theta)^2\cos^2\theta} & \pi \leq \theta \leq 2\pi \end{cases} \tag{2}$$

is a parametrization of a unit circle.

Solution. Since $g(\theta) \leq 1$ then w is real, and $x(\theta)^2 + w(\theta)^2 = 1$. Hence all the points $(x(\theta), w(\theta))$ lie on the unit circle. Now, $x(0) = 1$ and $x(\pi) = -1$, since $g(0) = g(\pi) = 1$. Since g is continuous x is continuous, and so $x([0,\pi]) = [-1, 1]$. It follows that $(x([0,\pi]), w([0,\pi]))$ is the upper half circle. By a similar argument $(x([\pi, 2\pi]), w([\pi, 2\pi]))$ is the lower half circle. □

6. Let $(x(\theta), w(\theta))$ be the parametrization of the unit circle C from (**??**). Again let

$$(x(\theta), y(\theta)) = (g(\theta)\cos\theta, g(\theta)\sin\theta)$$

be a parametrization of ∂D. Using (**??**), show that

$$\text{area}(D) + \pi = \oint_0^{2\pi} (x(\theta), -w(\theta)) \cdot (y'(\theta), x'(\theta))\,\mathrm{d}\theta.$$

Solution.

$$\begin{aligned}
\operatorname{area}(D) + 2\pi &= \oint_{\partial D} (0, x) \cdot \mathrm{d}\mathbf{s} + \oint_C (-y, 0) \cdot \mathrm{d}\mathbf{s} \\
&= \oint_0^{2\pi} x(\theta) y'(\theta)\,\mathrm{d}\theta - \oint_0^{2\pi} w(\theta) x'(\theta)\,\mathrm{d}\theta \\
&= \oint_0^{2\pi} x(\theta) y'(\theta) - w(\theta) x'(\theta)\,\mathrm{d}\theta \\
&= \oint_0^{2\pi} (x(\theta), -w(\theta)) \cdot (y'(\theta), y'(\theta))\,\mathrm{d}\theta.
\end{aligned}$$

□

7. Explain why it follows from the previous question that

$$\operatorname{area}(D) + \pi \leq \oint_0^{2\pi} \sqrt{(x(\theta)^2 + w(\theta)^2) \cdot (x'(\theta)^2 + y'(\theta)^2)}\,\mathrm{d}\theta.$$

Solution. The follows immediately from the Cauchy-Schwarz-Bunyakowski inequality for vectors in $\mathbb{R}^2$. □

8. Explain why

$$\operatorname{area}(D) + \pi \leq \operatorname{length}(\partial D).$$

Solution. Since $(x(\theta), w(\theta))$ is a parametrization of a unit circle, $x(\theta)^2 + w(\theta)^2 = 1$. The remaining integral is the length of ∂D. □

9. Recall the AMGM inequality: for $a, b > 0$ it holds that $\sqrt{ab} \leq (a+b)/2$. Use this to show that

$$\sqrt{4\pi \cdot \operatorname{area}(D)} \leq \operatorname{length}(\partial D).$$

For which shape are these two quantities equal?

Solution. Since

$$\operatorname{area}(D) + \pi \leq \operatorname{length}(\partial D),$$

$$\frac{2\operatorname{area}(D) + 2\pi}{2} \leq \operatorname{length}(\partial D).$$

By the AMGM inequality

$$\sqrt{2 \cdot \operatorname{area}(D) \cdot 2\pi} \leq \frac{2\operatorname{area}(D) + 2\pi}{2},$$

and so

$$\sqrt{4\pi \cdot \text{area}(D)} \leq \text{length}(\partial D).$$

Equality is achieved for circles. □