

18.022 Recitation Handout (with solutions)
03 December 2014

1. Find the flux of the vector field $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j}$ across the surface

$$z = 1 - x^2 - y^2, \quad z \geq 0.$$

Solution. A normal to the graph of $f(x, y)$ is $(-f_x, -f_y, 1)$, which is $(2x, 2y, 1)$. Normalizing this vector gives $\frac{1}{\sqrt{1+4x^2+4y^2}}(2x, 2y, 1)$. So the flux across the surface X is

$$\iint_X (x^2, xy, 0) \cdot (2x, 2y, 1) / \sqrt{1 + 4x^2 + 4y^2} dS = \iint_X \frac{2x(x^2 + y^2)}{\sqrt{1 + 4x^2 + 4y^2}} dS = 0,$$

since the integrand is an odd function of x and X is symmetric about the y - z plane. □

2. Write the surface $(s + t, s^2 + t^2, 3st(s + t))$ (for $(s, t) \in \mathbb{R}^2$) as the graph of a function $f(x, y)$.

Solution. Squaring the x -coordinate and subtracting the y -coordinate from the result gives $2st$.

Therefore $3st(s + t) = 3\left(\frac{x^2 - y}{2}\right)x = \boxed{\frac{3x(x^2 - y)}{2}}$. □

3. Calculate $\int_{\partial D} xy dS$, where $D = [0, 1]^3$.

Solution. We integrate over each of the six faces. The bottom face $[0, 1] \times [0, 1] \times \{0\}$ has integral

$$\int_0^1 \int_0^1 xy dx dy = 1/4.$$

The top face is the same. The faces $[0, 1] \times \{0\} \times [0, 1]$ and $\{0\} \times [0, 1] \times [0, 1]$ have zero contribution since the integrand vanishes on them. The faces $[0, 1] \times \{1\} \times [0, 1]$ and $\{1\} \times [0, 1] \times [0, 1]$ each have contribution

$$\int_0^1 \int_0^1 (y)(1) dx dy = \int_0^1 \int_0^1 (x)(1) dx dy = 1/2.$$

The sum of these contributions is $1/4 + 1/4 + 1/2 + 1/2 = \boxed{3/2}$. □

4. (Fun/Challenge problem, 7.2.25 in *Colley*) Let a be some positive constant. Consider the surface defined by $\mathbf{X}(s, t) = (x(s, t), y(s, t), z(s, t))$, where

$$\begin{aligned} x(s, t) &= \left(a + \cos \frac{s}{2} \sin t - \sin \frac{s}{2} \sin 2t\right) \cos s, \\ y(s, t) &= \left(a + \cos \frac{s}{2} \sin t - \sin \frac{s}{2} \sin 2t\right) \sin s, \\ z(s, t) &= \sin \frac{s}{2} \sin t + \cos \frac{s}{2} \sin 2t, \end{aligned}$$

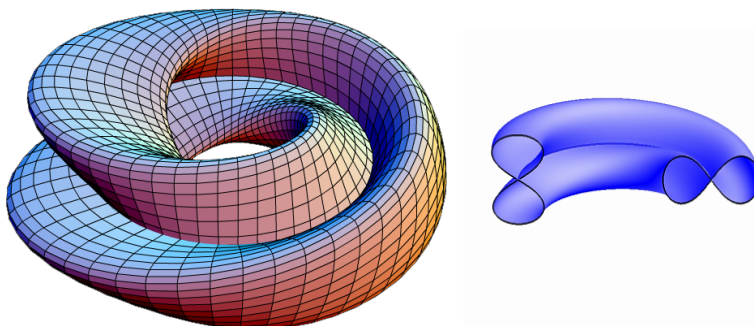
and s and t each vary over $[0, 2\pi]$.

(a) Describe the s -coordinate curve at $t = 0$.

(b) Calculate the standard normal vector \mathbf{N} along the s -coordinate curve at $t = 0$. In other words, find $\mathbf{N}(s, 0)$.

(c) Note that $\mathbf{X}(0, 0) = \mathbf{X}(2\pi, 0)$. Compare $\mathbf{N}(0, 0)$ and $\mathbf{N}(2\pi, 0)$. What can you conclude about the surface?

Solution. Below is a picture of the surface (left figure), which incidentally is called the *Klein bagel* and is a variant of the *Klein bottle*. It's the surface swept out by a figure 8 being moved around a circle in the x - y plane and twisted once along the way (see the figure to the right for what this process looks like halfway through). Thanks to Wikipedia for the pictures.



(a) The parametrization is $(a \cos s, a \sin s, 0)$, which is a circle of radius a centered at the origin. (b) The normal vector is $\mathbf{T}_s \times \mathbf{T}_t$, which when $t = 0$ simplifies to

$$\mathbf{N}(s, 0) = (a \cos s(2 \cos(s/2) + \sin(s/2)), a \sin s(2 \cos(s/2) + \sin(s/2)), a(2 \sin(s/2) - \cos(s/2))).$$

(c) We calculate $\mathbf{N}(0, 0) = (2a, 0, -a)$ and $\mathbf{N}(2\pi, 0) = (-2a, 0, a)$. Since these normal vectors are not the same, the surface is not orientable. See Section 7.2 in *Colley* for more discussion of this point. \square