18.022 Recitation Handout (with solutions)
01 December 2014

1. (7.1.30 in *Colley*, 4th edition) Let S be the surface defined by

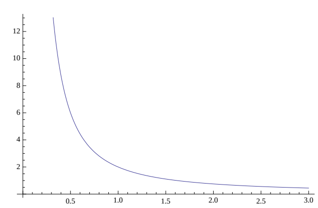
$$z = \frac{1}{\sqrt{x^2+y^2}} \text{ for } z \geq 1.$$

(a) Sketch the graph of this surface.

(b) Show that the volume of the region bounded by S and the plane $z = 1$ is finite. (You will need to use an improper integral.)

(c) Show that the surface area of S is infinite.

Solution. (a) See the graph below.



(b) The volume is given by $\int_0^1 \int_0^{2\pi} (r^{-1} - 1) r \, d\theta \, dr = \int_0^1 \int_0^{2\pi} (1 - r) \, d\theta \, dr$, which is finite since the integrand is bounded and the region of integration is compact. (c) The surface area of S is given by

$$\begin{aligned}\int_0^1 \int_0^{2\pi} \sqrt{1 + f_x^2 + f_y^2} \, r \, d\theta \, dr &= \int_0^1 \int_0^{2\pi} \sqrt{1 + x^2/(x^2+y^2)^3 + y^2/(x^2+y^2)^3} \, r \, d\theta \, dr \\ &= \int_0^1 \int_0^{2\pi} \sqrt{1 + r^{-4}} \, r \, d\theta \, dr \\ &= 2\pi \int_0^1 \sqrt{r^2 + r^{-2}} \, dr.\end{aligned}$$

This integral is infinite because the second term dominates, and $\int_0^1 \frac{dr}{r} = +\infty$. To prove this rigorously, we can drop the first term:

$$2\pi \int_0^1 \sqrt{r^2 + r^{-2}} \, dr \geq 2\pi \int_0^1 \sqrt{r^{-2}} \, dr = +\infty. \qquad \square$$

2. (7.2.1 in *Colley*, 4th edition) Let $\mathbf{X}(s,t) = (s, s+t, t)$, $0 \leq s \leq 1$, $0 \leq t \leq 2$. Find $\iint_{\mathbf{X}} (x^2 + y^2 + z^2) dS$.

Solution. We have

$$\int_0^1 \int_0^2 (s^2 + (s+t)^2 + t^2) \overbrace{\sqrt{\left(\frac{\partial(x,y)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2}}^{\sqrt{3}} \, dt\, ds = \boxed{26\sqrt{3}/3}. \qquad \square$$

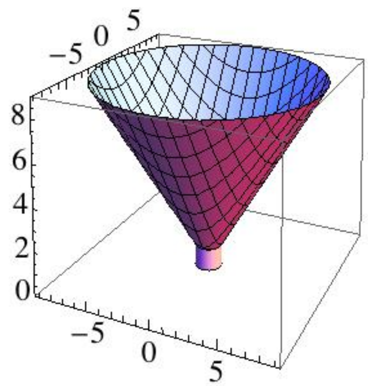
3. (7.2.27 in *Colley*, 4th edition) Let S be the funnel-shaped surface defined by $x^2 + y^2 = z^2$ for $1 \le z \le 9$ and $x^2 + y^2 = 1$ for $0 \le z \le 1$.

(a) Sketch S.

(b) Determine outward-pointing unit normal vectors to S.

(c) Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ and S is oriented by outward normals.

Solution. (a) See the graph below.



(b) The outward pointing unit vectors on the cylindrical part of the cylinder are $x\mathbf{i} + y\mathbf{j}$. On the lateral face, the unit normal is $\left(\frac{x}{z\sqrt{2}}, \frac{y}{z\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. (c) We calculate over the cylindrical surface S_1

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} (-y, x, z) \cdot (x, y, 0)\, dS = \iint_{S_1} 0\, dS = 0.$$

Over the conical surface, we calculate

$$\begin{aligned}\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} &= \iint_{S_1} (-y, x, z) \cdot \left(\frac{x}{z\sqrt{2}}, \frac{y}{z\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) dS \\ &= -\frac{1}{\sqrt{2}} \iint_{S_1} z\, dS.\end{aligned}$$

To evaluate this surface integral, we use slices perpendicular to the z-axis. The slice at height z is a thin circular strip of radius z and width $z\sqrt{2}$ (the factor of $\sqrt{2}$ arising from the 45° lean). Thus

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\begin{aligned}
-\frac{1}{\sqrt{2}}\iint_{S_1} z\,dS &= -\frac{1}{\sqrt{2}}\int_1^9 2\pi z(z\sqrt{2})\,dz \\
&= 2\pi\left[z^3/3\right]_1^9 \\
&= -1456\pi/3.
\end{aligned}
$$

Adding the contributions from S_1 and S_2, we get $\boxed{-1456\pi/3}$. □