18.022 Recitation Handout (with solutions)
24 November 2014

1. According to Coulomb's law, the force between a particle of charge q_1 at the origin and a particle of charge q_2 at the point $\mathbf{r} = (x, y, z) \in \mathbb{R}^3$ is given by

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{\mathbf{r}}{|\mathbf{r}|^3},$$

where ε_0 is a physical constant.

(a) Is $\mathbf{F}$ a conservative vector field? If so, find a function $\phi : \mathbb{R}^3 \to \mathbb{R}$ such that $\nabla\phi = \mathbf{F}$.

(b) If the distance between two charges is tripled, by what factor is the force between them reduced?

(c) How much work is required to move the second particle along the path

$$\gamma(t) = (1 + (1-t)\cos(t^2),\ \sqrt{\sin \pi t}, 4t - t^2) \qquad 0 \le t \le 1?$$

Express your answer in terms of q_1, q_2, and ε_0.

Solution. (a) Writing $\mathbf{F}$ as

$$\frac{q_1 q_2}{4\pi\varepsilon_0}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}}\right),$$

we see that $\mathbf{F}$ is the gradient of

$$\phi(x, y, z) = -\frac{q_1 q_2}{4\pi\varepsilon_0} \frac{1}{\sqrt{x^2+y^2+z^2}} = \boxed{-\frac{q_1 q_2}{4\pi\varepsilon_0} \frac{1}{|\mathbf{r}|}}.$$

Therefore, $\mathbf{F}$ is conservative.

(b) The magnitude of $\mathbf{F}$ is proportional to $|\mathbf{r}|/|\mathbf{r}|^3 = |\mathbf{r}|^{-2}$, so tripling the distance decreases the force by a factor of $\boxed{9}$.

(c) The amount of work required to move the particle from a point $\mathbf{r}_1$ to a point $\mathbf{r}_2$ along a path γ is $\int_\gamma \mathbf{F} \cdot d\mathbf{s}$. Since $\mathbf{F} = \nabla\phi$ is conservative, the value of this integral is $\phi(\mathbf{r}_2) - \phi(\mathbf{r}_1)$ no matter what path γ from $\mathbf{r}_1$ to $\mathbf{r}_2$ is chosen. For the given path, the starting point is $\gamma(0) = (2, 0, 0)$ and the ending point is $(1, 0, 3)$. Thus the work is $\boxed{\frac{q_1 q_2}{4\pi\varepsilon_0}\left(\frac{1}{2} - \frac{1}{\sqrt{10}}\right)}$. □

2. (6.2.23 in *Colley*) Let D be a region to which Green's theorem applies and suppose that $u(x, y)$ and $v(x, y)$ are two functions of class C^2 whose domains include D. Show that

$$\iint_D \frac{\partial(u, v)}{\partial(x, y)}\, dA = \oint_C (u\nabla v) \cdot d\mathbf{s},$$

where $C = \partial D$ is oriented as in Green's theorem.

Solution. Writing out the right-hand side and applying Green's theorem, we get

$$\begin{aligned}
\oint_C (u\nabla v) \cdot d\mathbf{s} &= \int uv_x\, dx + uv_y\, dy \\
&= \iint_D \frac{\partial}{\partial x}(uv_y) - \frac{\partial}{\partial y}(uv_x)\, dA \\
&= \iint_D u_x v_y - u_y v_x\, dA \\
&= \iint_D \frac{\partial(u, v)}{\partial(x, y)}\, dA.
\end{aligned}$$

□

3. (6.1.29 in *Colley*) Let C be a level set of the function $f(x, y)$. Show that $\int_C \nabla f \cdot d\mathbf{s} = 0$.

Solution. Letting a and b be the endpoints of the curve C, we calculate $\int_C \nabla f \cdot d\mathbf{s} = f(b) - f(a) = 0$, since f is constant on C. □