18.022 Recitation Handout (with solutions)
19 November 2014

1. (Open Courseware, 18.022 Fall 2010, Homework #12) Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field given by $\mathbf{F}(x, y, z) = ay^2\mathbf{i} + 2y(x + z)\mathbf{j} + (by^2 + z^2)\mathbf{k}$.

(a) For which values of a and b is the vector field $\mathbf{F}$ conservative?

(b) Find a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$ for these values.

(c) Find an equation describing a surface S with the property that for every smooth oriented curve C lying on S,

$$\int_C \mathbf{F} \cdot d\mathbf{s} = 0,$$

for these values.

Solution. (a) We calculate the curl of $\mathbf{F}$ to determine which values of a and b make $\mathbf{F}$ curl-free. The curl is $\nabla \times \mathbf{F} = (2by - 2y)\mathbf{i} + (2y - 2ay)\mathbf{k}$, which vanishes for all y when $\boxed{a = 1 \text{ and } b = 1}$.

(b) Since $f_x(x, y, z) = y^2$, we have $f(x, y, z) = xy^2 + g(y, z)$ for some function g. Differentiating with respect to y, we find that $f_y(x, y, z) = 2xy + g_y = 2xy + 2yz$, so $g(y, z) = y^2 z + h(z)$ for some function h. Differentiating with respect to z, we find that $f_z(x, y, z) = y^2 + h'(z) = y^2 + z^2$, which implies $h(z) = z^3/3$. Therefore $f(x, y, z) = xy^2 + y^2 z + z^3/3 + \text{(constant)}$.

(c) Since F is conservative, $\int_C \mathbf{F} \cdot d\mathbf{s} = f(b) - f(a)$. If this expression vanishes for all paths lying in S, then S is a level surface for f. Thus all such surfaces S may be found by choosing a possible value c of f and setting

$$S = \left\{(x, y, z) \in \mathbb{R}^3 \ : f(x, y, z) = c\right\}. \qquad \square$$

2. Find the area of the rectangle $D = [0, a] \times [0, b]$ using Green's theorem.

Solution. We calculate

$$\begin{aligned} 2\,\text{area}(D) &= \oint_{\partial D} -y\,dx + x\,dy \\ &= \int_0^a -0\,dx + \int_a^0 -b\,dx + \int_0^b a\,dy + \int_b^0 0\,dy \\ &= 2ab, \end{aligned}$$

so $\text{area}(D) = \boxed{ab}$. $\square$

3. (6.3.19 in *Colley*) Show that the line integral

$$\int_C \frac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}}$$

is path-independent, and evaluate it along the semicircular arc from $(2, 0)$ to $(-2, 0)$.

Solution. The integral is path independent because the vector field

$$\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$$

is the gradient of $\sqrt{x^2+y^2}$ and is therefore conservative. To evaluate the integral along any path from $(2, 0)$ to $(-2, 0)$, we just evaluate $\sqrt{x^2+y^2}$ at $(2, 0)$ and $(-2, 0)$ and subtract:

$$\sqrt{(-2)^2+0^2} - \sqrt{2^2+0^2} = \boxed{0}. \qquad \square$$