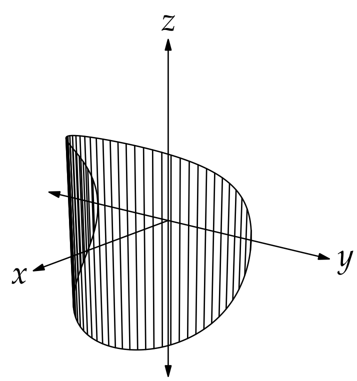
18.022 Recitation Handout (with solutions)
17 November 2014

1. Consider the surface $S = \left\{(x, y, z) \in \mathbb{R}^3 : x > 0 \text{ and } r = 1 \text{ and } -\sqrt{\frac{\pi^2}{4} - \theta^2} \leq z \leq \sqrt{\frac{\pi^2}{4} - \theta^2}\right\}$, shown below. (Note that r and θ refer to cylindrical coordinates.)

(a) Find the surface area of S using a scalar line integral.

(b) Check your answer by finding a non-calculus method of calculating the area of S.



Solution. (a) We integrate $f(x, y) = 2\sqrt{\frac{\pi^2}{4} - \theta^2}$ along the semicircular arc C in the xy-plane from $(0, -1, 0)$ to $(0, 1, 0)$. We note that for the path $\mathbf{x}(\theta) = (\cos\theta, \sin\theta, 0)$, we have $\|\mathbf{x}'(\theta)\| d\theta = d\theta$. So

$$\int_C f\, ds = \int_{-\pi/2}^{\pi/2} 2\sqrt{\frac{\pi^2}{4} - \theta^2}\, d\theta = \boxed{\pi^3/4}.$$

(b) S is just a disk wrapped around a cylinder. The area of the circle is $\pi r^2 = \pi(\pi/2)^2 = \pi^3/4$. □

2. In this problem, we discover a curl-free vector field which is not conservative.

(a) Define the vector field $\mathbf{F}(x, y, z) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0\right)$. Show that $\nabla \times \mathbf{F} = \mathbf{0}$.

(b) Show that the line integral of $\mathbf{F}$ around the origin-centered unit circle in the x-y plane does not vanish.

(c) How do you reconcile parts (a) and (b)?

Solution. (a) We calculate $\nabla \times \mathbf{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k} = \frac{x^2 + y^2 - x(2x) + (x^2 + y^2) - y(2y)}{(x^2 + y^2)^2}\mathbf{k} = \mathbf{0}$.

(b) The integral of **F** around the unit circle is

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} (-\sin\theta, \cos\theta) \cdot (-\sin\theta, \cos\theta)\, d\theta = \int_0^{2\pi} 1\, d\theta = 2\pi \neq 0.$$

(c) The vector field **F** is curl-free but not conservative. This does not contradict the fact that

a vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is curl-free if and only if it's conservative,

because **F** is defined on $D = \mathbb{R}^3 \setminus \{z\text{-axis}\}$, not on all of $\mathbb{R}^3$. In fact, the existence of curl-free nonconservative vector fields on D requires that D not be *simply connected*, which means that there exist loops in D which cannot be contracted to a point within D. In the present case, any loop which surrounds the z-axis has this property.

This observation serves as a gateway to a very important theory in differential geometry called *de Rham cohomology*. One generalizes vector fields to *differential forms* and the properties *curl-free* and *conservative* to *closed* and *exact* and uses information about closed nonexact forms on D to figure out things about the "holes" in the domain D. This is helpful because D might live in some high dimensional Euclidean space $\mathbb{R}^n$ for which visualizing the shape of D is difficult. □