18.022 Recitation Handout (with solutions)
13 November 2014

1. (5.5.10 in *Colley*) Evaluate the integral $\int_0^2 \int_{x/2}^{x/2+1} x^5(2y-x)e^{(2y-x)^2}\,dy\,dx$ by making the substitution $u = x$ and $v = 2y - x$.

Solution. Substitution shows that the limits of integration become $u \in [0,2]$ and $v \in [0,2]$. We calculate

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} 1 & 1/2 \\ 0 & 1/2 \end{vmatrix} = 1/2,$$

so the formula for change of variables gives

$$\begin{aligned}\int_0^2 \int_0^2 u^5 v e^{v^2} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du &= \int_0^2 \frac{1}{2}\left[\frac{1}{2}u^5 e^{v^2}\right]_0^2 du \\ &= \int_0^2 \frac{1}{4}u^5(e^4-1)\,du = \boxed{\frac{8}{3}(e^4-1)}.\end{aligned}$$

□

2. Let D be a parallelogram with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(2,1)$. Calculate $\iint_D 1\,dA$ in two ways:

(a) Find $\iint_D 1\,dA$ without using calculus.

(b) Find $\iint_D 1\,dA$ using the change of variables $u = 2x - 2y$ and $v = 2y$.

Solution. (a) The integral in question is the area of the parallelogram, which is (base)(height) = $1 \times 1 = \boxed{1}$. (b) To calculate this area using the suggested change of variables, we note that the linear transformation sends the four vertices of D to the vertices of a square $[0,2] \times [0,2]$. Since the transformation is linear, it sends D to $[0,2] \times [0,2]$. We calculate

$$\begin{aligned}\iint_D 1\,dA &= \int_0^2 \int_0^2 1 \begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix} du\,dv \\ &= \int_0^2 \int_0^2 1 \begin{vmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1/2 \end{vmatrix} du\,dv \\ &= 4(1/2)^2 = \boxed{1}.\end{aligned}$$

□

3. (5.5.30 in *Colley*) Find the volume of the solid that is bounded by the paraboloid $z = 9 - x^2 - y^2$, the xy-plane, and the cylinder $x^2 + y^2 = 4$.

Solution. To find the volume of the region, we integrate 1 over the region. We use cylindrical coordinates:

$$\text{volume} = \int_0^{2\pi} \int_0^2 \int_0^{9-r^2} 1\,r\,dz\,dr\,d\theta = (2\pi)\int_0^2 (9-r^2)r\,dr. = \boxed{28\pi}.$$

□

4. (5.5.29 in *Colley*) Find the volume of the region W that represents the intersection of the solid cylinder $x^2 + y^2 \leq 1$ and the solid ellipsoid $2(x^2 + y^2) + z^2 \leq 10$.

Solution. We integrate in cylindrical coordinates:

$$\int_0^1 \int_0^{2\pi} \int_{-\sqrt{10-2r^2}}^{\sqrt{10-2r^2}} r\,dz\,dr\,d\theta = 2\pi \int_0^1 2r\sqrt{10-2r^2}\,dr = \boxed{\frac{4}{3}\sqrt{2}\left(5\sqrt{5}-8\right)\pi}.$$

□