18.022 Recitation Handout (with solutions)
5 November 2014

1. Rewrite $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\,dz\,dy\,dx$ using the order $dx\,dy\,dz$.

Solution. $\int_0^1 \int_0^{1-z} \int_0^{y^2} f(x,y,z)\,dx\,dy\,dz$.

2. (5.4.15 in *Colley*) Integrate $f(x,y,z) = 1 - z^2$ over the tetrahedron W with vertices at the origin, $(1,0,0)$, $(0,2,0)$, and $(0,0,3)$.

Solution. Since the function depends only on z, it is convenient to set up the iterated integral with z varying first. We can solve for the equation of the plane passing through the points $(1,0,0)$, $(0,2,0)$, and $(0,0,3)$ by substituting into $Ax + By + Cz = 1$ and solving for A, B, and C. The integral becomes

$$\int_0^3 \int_0^{\frac{2}{3}(3-z)} \int_0^{1-y/2-z/3} (1-z^2)\,dx\,dy\,dz = \int_0^3 (1-z^2)\left(\left(\frac{3-z}{3}\right)\left(\frac{2(3-z)}{3}\right) - \frac{1}{4}\left[\frac{2}{3}(3-z)\right]^2\right) dz = \boxed{1/10}.$$

3. (5.8.19 in *Colley*) Set up a quadruple integral that computes the volume of the sphere $\{w^2+x^2+y^2+z^2 \le 1\}$ in $\mathbb{R}^4$.

Solution. By analogy with integrals for spheres in $\mathbb{R}^2$ and $\mathbb{R}^3$, we write

$$\boxed{\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \int_{-\sqrt{1-x^2-y^2-z^2}}^{\sqrt{1-x^2-y^2-z^2}} 1\,dw\,dz\,dy\,dx.}$$

4. (Fun/Challenge problem) For $(x,y) \neq (0,0)$, we define

$$f(x,y) = \frac{xy(x^2-y^2)}{(x^2+y^2)^3}.$$

Calculate the iterated integrals of f over $[0,2] \times [0,1]$.

Solution. See http://www.math.jhu.edu/~jmb/note/nofub.pdf