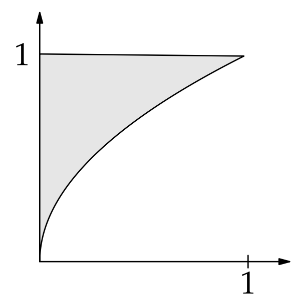
18.022 Recitation Handout (with solutions)
3 November 2014

1. Evaluate $\int_0^1 \int_0^{y^2} x^2 y\, dx\, dy$ and sketch the region of integration in $\mathbb{R}^2$ indicated by the limits of integration.

Solution. The domain of integration is shown below.



Evaluating the integral, we find

$$\int_0^1 \int_0^{y^2} x^2 y\, dx\, dy = \int_0^1 \left[\frac{x^3}{3} y \right]_0^{y^2} dy$$
$$= \int_0^1 \frac{y^7}{3}\, dy$$
$$= \boxed{1/24}.$$

2. Evaluate $\int_0^{\pi} \int_y^{\pi} \frac{\sin x}{x}\, dx\, dy$.

Solution. We reverse the order of integration. The region of integration is a triangle with vertices at $(0,0)$, $(\pi,0)$, and (π,π). So the integral is equal to

$$\int_0^{\pi} \int_0^{x} \frac{\sin x}{x}\, dy\, dx = \int_0^{\pi} [y]_0^x \sin(x)/x\, dx = \int_0^{\pi} \sin x\, dx = \boxed{2}.$$

3. (Putnam exam '89) Evaluate $\int_0^a \int_0^b e^{\max\{b^2 x^2, a^2 y^2\}}\, dy\, dx$ where a and b are positive. *Solution.* Omitted.

4. (Fun/Challenge, based on 5.2.29 in *Colley*) Define a function $f(x,y)$ on $[0,1] \times [0,2]$ by

$$f(x,y) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational and } y \le 1 \\ 2 & \text{if } x \text{ is irrational and } y > 1. \end{cases}$$

Show that the iterated Riemann integral $\int_0^1 \int_0^2 f(x,y)\,dy\,dx$ exists, and find its value. Show that the iterated Riemann integral $\int_0^2 \int_0^1 f(x,y)\,dx\,dy$ does not exist.

Solution. To show that $\int_0^1 \int_0^2 f(x,y)\,dy\,dx$ exists, we first calculate $\int_0^2 f(x,y)\,dy$. If x is rational, then this integral is $\int_0^2 1\,dy = 2$. If x is irrational, then this integral reduces to $\int_1^2 2\,dy = 2$. Therefore, regardless of the value of x, the inner integral $\int_0^2 f(x,y)\,dy$ is equal to 2. Therefore, $\int_0^1 \int_0^2 f(x,y)\,dy\,dx = \int_0^1 2\,dx = 2$.

On the other hand, there is no value of y for which the integral $\int_0^1 f(x,y)\,dx$ exists. To see this, note that (for $y \le 1$, say) the upper and lower Riemann sums are equal to 1 and 0, respectively. For if $0 = x_0 < x_1 < \ldots < x_n = 1$ is some partition of $[0,1]$, then the sum

$$\sum_{k=0}^{n-1} f(x_k^*)(x_{k+1} - x_k)$$

can be made as large as 1 by choosing each $x_k^* \in (x_{k+1} - x_k)$ to be rational and as small as 0 by choosing each $x_k^* \in (x_{k+1} - x_k)$ to be irrational. Since the inner integral doesn't exist, the iterated integral doesn't exist either.