18.022 Recitation Handout (with solutions)
29 October 2014

1. Minimize the function $f(x,y) = (x-y)^2$ subject to the constraint $xy = 1$ without using Lagrange multipliers. Verify that the method of Lagrange multipliers gives the same result.

Solution. Geometrically, we are asked to minimize the (squared) difference between x and y for a point on the graph of the function $y = \frac{1}{x}$. In fact, it is possible for $x - y$ to be zero, namely if $x = \pm 1$. Therefore, the minimum is zero, achieved at $(1,1)$ and $(-1,-1)$.

The method of Lagrange multipliers tells us that extrema occur at solutions to the system $\nabla f = \lambda \nabla g$, where $g(x,y) = xy$. Differentiating, we obtain

$$\begin{cases} 1 = xy \\ 2(x-y) = \lambda y \\ -2(x-y) = \lambda x. \end{cases}$$

Adding the second and third equations tells us that either $\lambda = 0$ or $x = y$. In the former case, the second equation implies $x = y$, so no matter what we have $x = y$. Substituting into the first equation gives $x = \pm 1$, as desired.

2. (4.3.19 in *Colley*) Find the maximum and minimum values of $f(x,y) = x^2 + xy + y^2$ on the closed disk $D = \{(x,y) : x^2 + y^2 \le 4\}$. Can you do it without using Lagrange multipliers? (Hint: $(x \pm y)^2 \ge 0$.)

Solution. We have $\nabla f = (2x + y, x + 2y)$ and $\nabla g = (2x, 2y)$. The system $\nabla f = \lambda \nabla g$ gives

$$\begin{cases} 2x + y = 2\lambda x \\ x + 2y = 2\lambda y. \end{cases}$$

Solving this system, we get either $x = y = 0$ or $2(\lambda - 1) = \pm 1$, i.e. $x = \pm y$. Suppose $x = y$ and $x^2 + y^2 \le 4$. Then $f(x,y)$ is maximized at $\pm(\sqrt{2}, \sqrt{2})$ with $f(x,y) = 6$ and minimized at $(x,y) = (0,0)$ with $f(x,y) = 0$. Now suppose $x = -y$ and $x^2 + y^2 \le 4$, $f(x,y)$ is maximized at $(-\sqrt{2}, \sqrt{2})$ and $(\sqrt{2}, -\sqrt{2})$ with $f(x,y) = 2$ and again minimized at $(x,y) = (0,0)$. Combining these two cases, we conclude that the maximum value and the minimum value of f are 6 and 0, respectively.

Without using Lagrange multipliers: Expanding $(x-y)^2 \ge 0$, we obtain $xy \le \frac{1}{2}(x^2 + y^2)$. Therefore

$$f(x,y) = x^2 + y^2 + xy \le \frac{3}{2}(x^2 + y^2) \le 6.$$

This means the value of $f(x,y)$ inside the disk D cannot exceed 6. On the other hand, we see that if $x = y = \sqrt{2}$ then $f(x,y) = 6$. So the maximum value of f is indeed 6. To find the minimum value of f, we start with $(x+y)^2 \ge 0$, or $xy \ge -\frac{1}{2}(x^2 + y^2)$. So

$$f(x,y) = x^2 + y^2 + xy \ge \frac{1}{2}(x^2 + y^2) \ge 0.$$

Finally $f(0,0) = 0$ justifies the minimum value of f.

3. (4.3.24 in *Colley*) Heron's formula for the area of a triangle whose sides have lengths x, y, and z is

$$\text{Area} = \sqrt{s(s-x)(s-y)(s-z)},$$

where $s = \frac{1}{2}(x + y + z)$ is the so-called semi-perimeter of the triangle. Use Heron's formula to show that, for a fixed perimeter P, the triangle with the largest area is equilateral.

Solution. Equivalently, we want to maximize the function

$$f(x, y, z) = (P - 2x)(P - 2y)(P - 2z)$$

subject to the constraint $g(x, y, z) = x + y + z = P$. Note that $\nabla g = (1, 1, 1)$ and

$$\nabla f = -2((P - 2y)(P - 2z), (P - 2x)(P - 2z), (P - 2x)(P - 2y)).$$

So, by the method of Lagrange multipliers, the maxima occur only when

$$(P - 2x)(P - 2y) = (P - 2y)(P - 2z) = (P - 2z)(P - 2x).$$

Since none of $(P-2x), (P-2y), (P-2z)$ can be zero (why?), the only case is when $P-2x = P-2y = P-2z$, i.e. $x = y = z$.