18.022 Recitation Handout (with solutions)
27 October 2014

1. Find the second order Taylor polynomial for $f(x,y) = \cos(x+2y)$ at the origin. What is the second order Taylor polynomial for $g(\theta) = \cos\theta$ at $\theta = 0$?

Solution. The second order Taylor polynomial is

$$f(0,0) + f_x(0,0)x + f_y(0,0)y + f_{xy}(0,0)xy + \frac{1}{2}f_{xx}(0,0)x^2 + \frac{1}{2}f_{yy}(0,0)y^2,$$

which equals

$$1 - 2xy - \frac{1}{2}x^2 + 2y^2.$$

This can also be obtained by substituting $x+2y$ into the Taylor polynomial $1 - \frac{1}{2}\theta^2$ for $\cos\theta$ at $\theta = 0$.

2. (a) Find the critical points of $f(x,y) = x^2 + 4xy + y^2$. Use the second derivative test for local extrema to determine whether the point is a local maximum, a local minimum, or a saddle point.

(b) Find the critical points of $g(x,y) = x^2 + xy + y^2$. Use the second derivative test for local extrema to determine whether the point is a local maximum, a local minimum, or a saddle point.

Solution. (a) The gradient of f is $(2x+4y, 4x+2y)$, which equals $\mathbf{0}$ if and only if $(x,y) = (0,0)$. Therefore, the origin is the only critical point of f. The Hessian evaluated at $(0,0)$ is

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} = 2\cdot 2 - 4\cdot 4 < 0,$$

so the origin is a saddle point.

(b) The gradient of g is $(2x+y, x+2y)$, which equals $\mathbf{0}$ if and only if $(x,y) = (0,0)$. Therefore, the origin is the only critical point of g. The Hessian evaluated at $(0,0)$ is

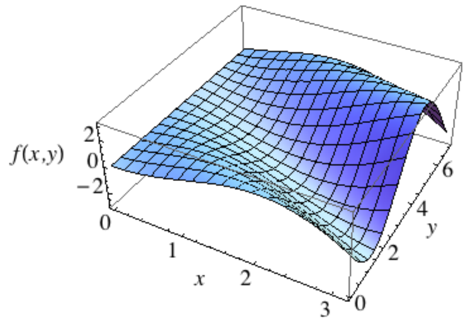
$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 2\cdot 2 - 1\cdot 1 > 0,$$

so the critical point is a local extremum. Since $f_{xx} > 0$, the Hessian is positive definite and the critical point is a local minimum.

3. (a) What theorem ensures that the function $f(x,y) = x\sin(x+y)$ defined on the rectangle $\{(x,y) : 0 \leq x \leq \pi, 0 \leq y \leq 7\}$ has an absolute maximum and an absolute minimum? Verify the hypotheses of that theorem.

(b) Find the absolute extrema of f. You are given that there are no absolute extrema on the top or bottom of the rectangle; see the surface plot below to guide your intuition.

Solution. (a) The extreme value theorem ensures that the function achieves absolute extrema, because it is continuous function defined on a compact (that is, closed and bounded) set.



(b) If f has an extremum in the interior of the rectangle, then $Df = \mathbf{0}$ there. Since $Df = (\sin(x+y) + x\cos(x+y), x\cos(x+y))$, there are no critical points in the interior of the rectangle. To see this, note that the second coordinate is zero if and only if $\cos(x+y) = 0$. If $\cos(x+y) = 0$, then the first coordinate is zero if and only if $\sin(x+y) = 0$. But sine and cosine never vanish simultaneously, so there are no critical points.

It follows that f has its absolute extremum on the edges or at one of the vertices of the rectangle. We look at each side one at a time.

- On the bottom side of the rectangle, $f(x,y) = f(x,0) = x\sin x$, which has a minimum of 0 at $(0,0)$ and $(\pi,0)$ and a maximum of about 1.81 at about $(2.02,0)$.
- On the top side of the rectangle, $f(x,y) = f(x,7) = x\sin(x+7)$, which has a minimum of $\pi\sin(\pi+7)$ at $(\pi,7)$ and a maximum of about 1.2 at about $(1.46,7)$.
- On the right side, $f(x,y) = f(\pi,y) = \pi\sin(\pi+y)$, which has minimum of $-\pi$ at $(\pi,\pi/2)$ and a maximum of π at $(\pi,3\pi/2)$.
- On the left side, $f(x,y) = f(0,y) = 0$.

Putting all this together, we see that the absolute maximum of $\boxed{\pi}$ is achieved at $\boxed{(\pi,3/\pi/2)}$, while the minimum of $\boxed{-\pi}$ is achieved at $\boxed{(\pi,\pi/2)}$.