

18.022 Recitation Handout (with solutions)
22 October 2014

1. Verify that divergence, curl, and gradient are linear operators.

Solution. For divergence, we want to show that for all vector fields \mathbf{F} and \mathbf{G} and scalars α and β , we have

$$\nabla \cdot (\alpha \mathbf{F} + \beta \mathbf{G}) = \alpha \nabla \cdot \mathbf{F} + \beta \nabla \cdot \mathbf{G}.$$

The left-hand side is

$$\frac{\partial}{\partial x}(\alpha F_1 + \beta G_1) + \frac{\partial}{\partial y}(\alpha F_2 + \beta G_2) + \frac{\partial}{\partial z}(\alpha F_3 + \beta G_3) = \alpha \frac{\partial F_1}{\partial x} + \beta \frac{\partial G_1}{\partial x} + \alpha \frac{\partial F_2}{\partial y} + \beta \frac{\partial G_2}{\partial y} + \alpha \frac{\partial F_3}{\partial z} + \beta \frac{\partial G_3}{\partial z},$$

which equals the right-hand side. Calculations for gradient and curl are similar.

2. Let $\mathbf{F}(x, y, z) = (3x^2 + \frac{1}{2}y^2 + e^z, xy + z, f(x, y, z))$. Find all f such that \mathbf{F} is curl-free.

Solution. The third component of $\nabla \times \mathbf{F}$ is $y - (1/2)(2y) = 0$, as desired. For the first component to be zero, we must have $f_y = 1$, and for the second component to be zero we must have $f_x = e^z$. Integrating these two equations tells us that $f(x, y, z) = y + C_1(x, z)$ and $f(x, y, z) = xe^z + C_2(y, z)$ for functions C_1 and C_2 which do not depend on y or x respectively. Putting these two together, we see that $f(x, y, z) = xe^z + y + C(z)$ for any differentiable function C .

3. Confirm that for a vector field $\mathbf{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, we have

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F},$$

where $\nabla^2 \mathbf{F}$ is defined to mean “take the Laplacian of each component of \mathbf{F} .” Is it possible to derive this identity from $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$?

Solution. [Omitted]

4. Let \mathbf{F} be a C^2 vector field on \mathbb{R}^3 . Show that $\nabla \times \mathbf{F}$ is incompressible.

Solution. We calculate

$$\nabla \cdot (\nabla \times \mathbf{F}) = \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} = 0,$$

since the mixed partials don't depend on the order of differentiation, as \mathbf{F} is C^2 .