18.022 Recitation Handout (with solutions)
20 October 2014

1. (3.2.17 in *Colley*) Use the formula

$$\kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}$$

to show that if f is C^2 on an interval $[a, b]$ then the curvature of the graph $y = f(x)$ is

$$\kappa = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}.$$

Solution. We parametrize the graph as usual by $\mathbf{x}(t) = (t, f(t))$ for $t \in [a, b]$. Then $\mathbf{v}(t) = (1, f'(t))$ and $\mathbf{a}(t) = (0, f''(t))$. Taking the norm of $\mathbf{v} \times \mathbf{a}$ gives $|f''(t)|$, and $\|v\|^{3/2} = (1 + f'(t)^2)^{3/2}$. Substituting into the given formula gives the desired expression for κ.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be a map defined by $f(\mathbf{x}) = (|\mathbf{x}|^2, 1, |\mathbf{x}|)$ for $\mathbf{x} \in \mathbb{R}^2$. Find the total derivative Df.

Solution. The total derivative is the matrix of partial derivatives, which is

$$Df(x_1, x_2) = \begin{pmatrix} 2x_1 & 2x_2 \\ 0 & 0 \\ \frac{x_1}{|x|} & \frac{x_2}{|x|} \end{pmatrix}.$$

3. Sketch the curve $\mathbf{x}(t) = (t \cos t, t \sin t)$ and find its unit tangent vector.

Solution. The unit tangent vector is given by $\mathbf{x}'(t)/|\mathbf{x}'(t)|$. We calculate

$$\mathbf{x}'(t) = (\cos t - t \sin t, \sin t + t \cos t),$$

the squared norm of which is

$$\cos^2 t - 2t \sin t \cos t + 2t \sin t \cos t + t^2 \sin^2 t + t^2 \cos^2 t = 1 + t^2.$$

So the unit tangent vector is

$$\left(\frac{\cos t - t \sin t}{\sqrt{1 + t^2}}, \frac{\sin t + t \cos t}{\sqrt{1 + t^2}} \right).$$

4. Let $f(x, y) = \log(x^2 + y^2)$ for $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Show that $\nabla \cdot (\nabla f) = 0$.

Solution. The gradient of f is

$$\nabla f = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right).$$

The divergence of this vector field is

$$\nabla \cdot (\nabla f) = \frac{\partial}{\partial x} \left[\frac{x}{x^2 + y^2} \right] + \frac{\partial}{\partial y} \left[\frac{y}{x^2 + y^2} \right] = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2} = 0.$$