18.022 Recitation Handout (with solutions)
6 October 2014

1. Let $f(x,y) = e^{3x+y}$, and suppose that $x = s^2 + t^2$ and $y = 2 + t$. Find $\partial f/\partial s$ and $\partial f/\partial t$ by substitution and by means of the chain rule. Verify that the results are the same for the two methods.

Solution. The chain rule gives

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} = 3e^{x+y}(2s) + e^{3x+y}\cdot 0 = 6se^{3x+y} = 6se^{3s^2+3t^2+2+t}.$$

Substitution gives $\frac{\partial}{\partial s}(e^{3s^2+3t^2+2+t}) = 6se^{3s^2+3t^2+2+t}$. The calculations for $\frac{\partial f}{\partial t}$ are similar.

2. A conical ice sculpture melts in such a way that its height decreases at a rate of 0.001 meters per second and its radius decreases at a rate of 0.002 meters per second. At what rate is the volume of the sculpture decreasing when its height reaches 3 meters, assuming that its radius is 2 meters at that time? Express your answer in terms of π and in units of cubic meters per second.

Solution. Since the volume V of a cone can be expressed in terms of its radius and height as $V = \frac{1}{3}\pi r^2 h$, the chain rule implies

$$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial V}{\partial h}\frac{\partial h}{\partial t} = \frac{1}{3}\pi\left(2rh\frac{\partial r}{\partial t} + r^2\frac{\partial h}{\partial t}\right).$$

Substituting the given derivatives values, we get $(\pi/3)(2\cdot 3\cdot 2\cdot 0.002 + 4\cdot 0.001) = 28\pi/3000 = \boxed{7\pi/750}$.

3. Given a nonzero vector $\mathbf{a} \in \mathbb{R}^n$, what unit vector $\mathbf{u} \in \mathbb{R}^n$ maximizes the dot product $\mathbf{a}\cdot\mathbf{u}$? What unit vector *minimizes* the dot product? Prove that these really are the maximum and minimum, and comment on how this observation relates to the gradient ∇f of a function $f : \mathbb{R}^n \to \mathbb{R}$.

Solution. Choosing $\mathbf{u} = \mathbf{a}/\|\mathbf{a}\|$ maximizes the dot product, and choosing $\mathbf{u} = -\mathbf{a}/\|\mathbf{a}\|$ minimizes the dot product. The Cauchy-Schwarz inequality ensures that these values are extremal. By the chain rule, the gradient ∇f has the property that the infinitesimal rate of increase of f in the direction $\mathbf{u}$ is given by $\nabla f \cdot \mathbf{u}$. Therefore, the direction of the gradient is also the direction of direction of f's greatest increase. Similarly, the direction of $-\nabla f$ is the direction of f's greatest decrease.

4. Consider the sphere S passing through the point $P = (1,2,3)$ and centered at the origin. Find the equation of the plane tangent to S at P.

Solution. The sphere S is the set of points for which $x^2 + y^2 + z^2 = 14$. If we define $f(x,y,z) = x^2 + y^2 + z^2$, then S is a level surface of f. The normal to the tangent plane of S at (x_0, y_0, z_0) is given by $(\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)\big|_{(x_0,y_0,z_0)}$. Differentiating and substituting gives $(2,4,6)$ for the normal vector. Substituting into the equation $\mathbf{n}\cdot((x,y,z) - P) = 0$ for the plane with normal $\mathbf{n}$ at the point P. In standard form, we get $\boxed{2x + 4y + 6z = 14}$.

5. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$. Is it possible for $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ to exist at $(0,0)$ while f is not differentiable at $(0,0)$? Prove that it isn't possible, or provide an example to show that it is possible.

Solution. $f(x,y) = xy/(x^2 + y^2)$ with $f(0,0) = 0$ is not continuous (and hence not differentiable) at $(0,0)$.

However, $f = 0$ on the union of the coordinate axes, so its partial derivatives are both defined and equal to 0.