18.022 Recitation Handout (with solutions)
24 September 2014

1. Let $A = \begin{pmatrix} 2 & 6 \\ 0 & 2 \end{pmatrix}$, and let U be the unit square $\{(x, y) : 0 \le x \le 1 \text{ and } 0 \le y \le 1\}$ in $\mathbb{R}^2$. Let U' be the image under A of U. Find the area of U.

Solution. Note that $A = 2\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} =: 2M$. The image of U under M is a parallelogram with unit base and height, and therefore it has unit area. The factor of 2 doubles the shape in both dimensions, giving a factor of 4 increase from the original area. So the area of U' is $1 \times 4 = \boxed{4}$.

2. Find the distance from the line $(4 + t, -1 - 2t, 3 - 7t)$ to the plane $3x - 2y + z = 3$.

Solution. Since $(3, -2, 1) \cdot (1, -2, -7) = 0$, the line is parallel to the plane. Let $P = (4, -1, 3)$ be a point on the line, and let Q be the point in the plane which is nearest to P. Since $\overrightarrow{QP}$ is parallel to the plane's normal vector $(3, -2, 1)$, we can write $Q = P - \lambda(3, -2, 1)$ for some scalar λ, substitute the resulting coordinates into the equation for the plane, and solve to find $\lambda = 1$. Therefore, the distance from the line to the plane is $\sqrt{3^2 + (-2)^2 + 1^2} = \boxed{\sqrt{14}}$.

3. Let $A = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 \\ 5 & 0 \end{pmatrix}$. Find $AB - BA$.

Solution. We calculate $AB = \begin{pmatrix} -13 & -4 \\ 21 & -2 \end{pmatrix}$ and $BA = \begin{pmatrix} 0 & -11 \\ 10 & -15 \end{pmatrix}$, so the difference $AB - BA$ is $\boxed{\begin{pmatrix} -13 & 7 \\ 11 & 13 \end{pmatrix}}$. Notice that this matrix measures the failure of A and B to commute.

4. Consider the function $f(x, y, z) = (x^2 + y^2)/\sin(z)$. Describe the level surfaces for different values. What coordinate system is best suited for this?

Solution. Cylindrical coordinates are best suited, since $x^2 + y^2$ simplifies to r^2. The level surfaces $\{(x, y, z) : f(x, y, z) = c\}$ are surfaces of revolution obtained by revolving a graph of $r^2 = c \sin z$ (thought of as a curve in 2D) about the z-axis.

5. We say that a function $f : \mathbb{R}^m \to \mathbb{R}^n$ is linear if $f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$. Characterize all linear functions from $\mathbb{R}$ to $\mathbb{R}$. Is $f(x) = 7x - 4$ linear, according to this definition?

Solution. Applying the definition of linearity with $x = 1$, $y = 0$, and $\lambda \in \mathbb{R}$ arbitrary, we find that $f(\lambda) = \lambda f(1)$. In other words, every linear function takes the form $f(x) = mx$ for some constant m. Conversely, every function of the form $f(x) = mx$ is linear. Therefore, the linear functions are the ones whose graphs are lines passing through the origin.