18.022 Recitation Handout (with solutions)
22 September 2014

1. (1.9.22 in *Colley*) Given an arbitrary tetrahedron, associate to each of its four triangular faces a vector outwardly normal to that face with length equal to the area of the face. Show that the sum of these four vectors is zero.

Solution. If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are three vectors along three edges of the tetrahedron, then the four vectors in question are $\frac{1}{2}\mathbf{a}\times\mathbf{b}$, $\frac{1}{2}\mathbf{b}\times\mathbf{c}$, $\frac{1}{2}\mathbf{c}\times\mathbf{a}$, and $\frac{1}{2}(\mathbf{c}-\mathbf{a})\times(\mathbf{b}-\mathbf{a})$. Summing these and using linearity of the cross product, we find that their sum vanishes.

2. (1.9.14 in *Colley*) The median of a triangle is the line segment that joins a vertex of a triangle to the midpoint of the opposite side. The purpose of this problem is to use vectors to show that the medians of a triangle all meet at a point.

(a) Let M_1 be the midpoint of BC, let M_2 be the midpoint of AC, and let M_3 be the midpoint of AB. Write $\overrightarrow{BM_2}$ and $\overrightarrow{CM_3}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

(b) Use the fact that $\overrightarrow{CB}=\overrightarrow{CP}+\overrightarrow{PB}=\overrightarrow{CA}+\overrightarrow{AB}$ to show that P must lie two-thirds of the way from B to M_2 and two-thirds of the way from C to M_3.

(c) Use part (b) to show why all three medians must meet at P.

Solution. (a) We have $\overrightarrow{CM_3}=\frac{1}{2}\overrightarrow{AB}-\overrightarrow{AC}$ and $\overrightarrow{BM_2}=\frac{1}{2}\overrightarrow{AC}-\overrightarrow{AB}$, by definition of vector addition. (b) Define λ and μ so that $\overrightarrow{CP}=\lambda\overrightarrow{CM_3}$ and $\overrightarrow{BP}=\overrightarrow{BM_2}$. Using the suggested equation, we get $\overrightarrow{AB}-\overrightarrow{AC}=(\lambda/2+\mu)\overrightarrow{AB}-(\lambda+\mu/2)\overrightarrow{AC}$. Since $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are linearly independent, this happens only if the coefficients on both sides match. Solving this system for λ and μ, we find $\lambda=\mu=2/3$ as desired. (c) If the segment from A to M_1 does not pass through P, then it intersects CM_3 and BM_2 at two different points Q and R. However, the preceding argument applied to the pairs (A,C) and (A,B) in place of (B,C) shows that Q and R each lie 2/3 of the way from A to M_1. Thus $Q=R$, a contradiction.

3. Find the equation of a plane P perpendicular to $(1,2,-1)$ containing the line that passes through the two points $(-2,5,4)$ and $(5,1,3)$. Find the distance from P to the plane P' whose equation is $x+2y-z=28$.

Solution. The equation of the plane is $(1,2,-1)\cdot(x+2,y-5,z-4)=0$, which simplifies to $x+2y-z=4$. Note that we would have gotten the same result had we used the second point instead of the first point. The distance between the planes is $(28-4)/\sqrt{1^2+2^2+(-1)^2}=\boxed{4\sqrt{6}}$. See the 19 September 2012 recitation handout for a derivation of this formula.

4. (Fun/Challenge problem) Example 9 of Section 1.5 in the book asks us to compute the distance between the lines $\ell_1(t)=(0,5,-1)+t(2,1,3)$ and $\ell_2(t)=(-1,2,0)+t(1,-1,0)$. The solution given in the book uses vectors; our goal here is to take an algebraic approach for comparison. Define $D(s,t)=|\ell_1(t)-\ell_2(t)|^2$, and find the values of s and t which minimize $D(s,t)$ (using calculus methods or otherwise).

Solution. We have $D(s,t)=(2t-(-1+s))^2+(5+t-(2-s))^2+(-1+3t)^2$, and if D is minimized then

$\partial D/\partial s = 0$ and $\partial D/\partial t = 0$. These two derivatives are $\partial D/\partial s = 4 + 4s - 2t$ and $\partial D/\partial t = 4 - 2s + 28t$. Setting each of these equal to 0 and solving the system, we find $s = -10/9$ and $t = -2/9$. Substituting into $D(s, t)$, we find that the minimal squared distance is $25/3$.