18.022 Recitation Handout (with solutions)
15 September 2014

1. Find the cosine of the angle between the vectors $(-4,2,6,5)$ and $(3,1,0,-7)$.

Solution. The cosine of the angle between two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined by $\cos\theta = \mathbf{a}\cdot\mathbf{b}/|\mathbf{a}||\mathbf{b}| = (-45/(9\cdot\sqrt{59}) = -5/\sqrt{59}$.

2. Is the cross-product associative? Is the dot product associative? Prove or give a counterexample for each.

Solution. The cross-product is not associative. For example, $(\mathbf{i}\times\mathbf{i})\times\mathbf{j} = 0$ and $\mathbf{i}\times(\mathbf{i}\times\mathbf{j}) = -\mathbf{j}$. The dot product is not associative because neither $(\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c}$ nor $\mathbf{a}\cdot(\mathbf{b}\cdot\mathbf{c})$ is defined.

3. Find an equation for the plane that contains the lines

$$\begin{cases} x(t) = 5+t \\ y(t) = 1-t \\ z(t) = 4-3t. \end{cases}$$

and

$$\begin{cases} x(t) = 5-4t \\ y(t) = 1+t \\ z(t) = 4-3t. \end{cases}$$

Solution. The plane contains the point $P_0 = (5,1,4)$ and consists of all the points $P = (x,y,z)$ for which $\overrightarrow{P_0P}\cdot(\mathbf{v_1}\times\mathbf{v_2}) = 0$, where $\mathbf{v}_1 = (1,-1,-3)$ and $\mathbf{v}_2 = (-4,1,-3)$. Since $\mathbf{v_1}\times\mathbf{v_2} = (6,15,-3)$, the equation of the plane simplifies to $\boxed{6x+15y-3z=-33}$.

4. Find the distance from the origin to the plane $x+y+z=1$.

Solution. The shortest line segment from the origin O to the plane $x+y+z=1$ intersects the plane at some point $P_0 = (x_0,y_0,z_0)$. Since $\overrightarrow{PO}$ is perpendicular to the plane, the line containing $\overrightarrow{PO}$ takes the form

$$\begin{cases} x(t) = x_0+t \\ y(t) = y_0+t \\ z(t) = z_0+t. \end{cases}$$

Setting each of these coordinates equal to 0, we see that $x_0 = y_0 = z_0$. Since the coordinates sum to 1, they are $(1/3,1/3,1/3)$. Thus the distance is $\sqrt{(1/3)^2+(1/3)^2+(1/3)^2} = \boxed{\sqrt{3}/3}$.

We remark that it is also possible to solve this problem by computing the volume of the tetrahedron with vertices at O, $(1,0,0)$, $(0,1,0)$, and $(0,0,1$ in two different ways.

5. Give a geometric description of what each of the following matrices does to a vector.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Solution. A does nothing, B increases the length by a factor of 3, C reflects across the y-axis, and D reflects across the line $y = x$.

6. What matrices have the following geometric descriptions?

(a) reflect across the x-axis

(b) reverse the direction of the vector and double its length

(c) halve the length of the vector (while preserving the direction)

(d) rotate a vector 90 degrees counterclockwise

(e) project a vector in $\mathbb{R}^3$ onto the x-y plane

Solution. (a) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, (b) $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$, (c) $\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$, (d) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, (e) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$,

7. The volume of a parallelepiped is the product of the area of its base and its height. Consider the parallelepiped spanned by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. You may suppose for simplicity that $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{a}$ form a right-handed triple of vectors, which means that a right-handed screw rotated an angle less than 180° from $\mathbf{b}$ to $\mathbf{c}$ advances in the direction of $\mathbf{a}$.

(a) Let us think of the parallelogram spanned by $\mathbf{b}$ and $\mathbf{c}$ as the base of the parallelepiped. What is the (signed) area of this parallelogram?

(b) What is the height of the parallelogram, in terms of $\mathbf{a}$ and the unit vector pointing in the direction of $\mathbf{b} \times \mathbf{c}$?

(c) Put together parts (a) and (b) to derive the triple scalar product formula for the volume of a parallelepiped.

Solution. The (signed) area of the parallelogram is $|\mathbf{b} \times \mathbf{c}|$. The height of the parallelogram is $\mathbf{a} \cdot \mathbf{u}$, where $\mathbf{u}$ is the unit vector in the direction of $\mathbf{b} \times \mathbf{c}$. Therefore, the volume of the parallelepiped is

$$\mathbf{a} \cdot (\mathbf{u}|\mathbf{b} \times \mathbf{c}|) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

8. (Fun/Challenge problem) It is possible to prove the vector triple product formula

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

in a tedious way using coordinates. This problem outlines a more conceptual proof, taken from a note written by William C. Schulz.

(a) Use the paralellepiped interpretation of the triple scalar product to show that

$$\mathbf{b} \cdot (\mathbf{c} \times \mathbf{n}) = \mathbf{c} \cdot (\mathbf{n} \times \mathbf{b}) = \mathbf{n} \cdot (\mathbf{b} \times \mathbf{c})$$

(b) Use the right-hand rule to observe that if $\mathbf{c}$ is perpendicular to $\mathbf{n}$, then

$$\mathbf{n} \times (\mathbf{c} \times \mathbf{n}) = |\mathbf{n}|^2 \mathbf{c}.$$

(c) Show that it suffices to consider the case where $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ form a basis for $\mathbb{R}^3$.

(d) Write $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ as a linear combination of $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$, so that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \kappa \mathbf{a} + \lambda \mathbf{b} + \mu \mathbf{c}, \tag{1}$$

(e) The easiest coefficient to determine is κ. What is it?

(f) To find λ, dot both sides of (**??**) with $\mathbf{c} \times \mathbf{n}$, where $\mathbf{n} := \mathbf{b} \times \mathbf{c}$.

(g) To find μ, dot both sides of (**??**) with $\mathbf{b} \times \mathbf{n}$, where $\mathbf{n} := \mathbf{b} \times \mathbf{c}$.

Solution. See http://mathdl.maa.org/images/cms_upload/0746834213321.di020720.02p0099b.pdf.