18.022 Recitation Handout (with solutions)
10 September 2014

1. For each of the following pairs of vectors $\mathbf{a}$ and $\mathbf{b}$, calculate $\mathbf{a}\cdot\mathbf{b}$ and $\|\mathbf{a}\|\,\|\mathbf{b}\|$.

(a) $\mathbf{a} = (1,5)$ and $\mathbf{b} = (-2,3)$

Solution. $\mathbf{a}\cdot\mathbf{b} = 1\cdot -2 + 5\cdot 3 = 13$, and $\|\mathbf{a}\|\,\|\mathbf{b}\| = \sqrt{1^2+5^2}\sqrt{(-2)^2+3^2} = \sqrt{26\cdot 13} = 13\sqrt{2}$

(b) $\mathbf{a} = (3,-5)$ and $\mathbf{b} = (2,0)$

Solution. $\mathbf{a}\cdot\mathbf{b} = 3\cdot 2 + -5\cdot 0 = 6$, and $\|\mathbf{a}\|\,\|\mathbf{b}\| = 2\sqrt{17}$

(c) $\mathbf{a} = (-2,4,1)$ and $\mathbf{b} = (4,1,2)$

Solution. $\mathbf{a}\cdot\mathbf{b} = -2\cdot 4 + 4\cdot 1 + 1\cdot 2 = -2$, and $\|\mathbf{a}\|\,\|\mathbf{b}\| = 21$.

(d) Conjecture an inequality relating $|\mathbf{a}\cdot\mathbf{b}|$ and $\|\mathbf{a}\|\,\|\mathbf{b}\|$ for $\mathbf{a},\mathbf{b}\in\mathbb{R}^n$.

Solution. In all three cases above, we see that $|\mathbf{a}\cdot\mathbf{b}| \le \|\mathbf{a}\|\,\|\mathbf{b}\|$.

(e) (Fun/Challenge problem) To prove the inequality conjectured in (d) (called the *Cauchy-Schwarz inequality*), expand the left-hand side of the inequality $\|\mathbf{a}+\lambda\mathbf{b}\|^2 \ge 0$, where λ is any real number.

Solution. Expanding $\|\mathbf{a}+\lambda\mathbf{b}\|^2 \ge 0$ gives $\|\mathbf{a}\|^2 + 2\lambda\mathbf{a}\cdot\mathbf{b} + \lambda^2\|\mathbf{b}\|^2 \ge 0$. Since we want to get the strongest possible inequality, we choose λ to minimize the quadratic expression on the left-hand side. Substituting $\lambda = -\mathbf{a}\cdot\mathbf{b}/\|\mathbf{b}\|^2$, the resulting inequality simplifies to $|\mathbf{a}\cdot\mathbf{b}| \le \|\mathbf{a}\|\,\|\mathbf{b}\|$.

2. (1.3.20 in *Colley*) Suppose that a force $\mathbf{F} = (1,-2)$ is acting on an object moving parallel to the vector $(4,1)$. Decompose $\mathbf{F}$ into a sum of vectors $\mathbf{F}_1$ and $\mathbf{F}_2$, where $\mathbf{F}_1$ points along the direction of motion and $\mathbf{F}_2$ is perpendicular to the direction of motion.

Solution. We project $(1,-2)$ onto $(4,1)$ using the formula $\mathrm{proj}_{\mathbf{b}}\mathbf{a} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b}$. We obtain $F_1 = \frac{2}{17}(4,1) = (\frac{8}{17},\frac{2}{17})$ and $F_2 = F_-F_2 = (1,-2) - \left(\frac{8}{17},\frac{2}{17}\right) = \left(\frac{9}{17},-\frac{36}{17}\right)$.

3. (1.3.17 in *Colley*) Is it ever the case that the projection of $\mathbf{a}$ onto $\mathbf{b}$ and the projection of $\mathbf{b}$ onto $\mathbf{a}$ are the same vector? If so, under what conditions?

Solution. Suppose $\mathrm{proj}_{\mathbf{b}}\mathbf{a} = \mathrm{proj}_{\mathbf{a}}\mathbf{b}$. Since a projection onto $\mathbf{a}$ is either 0 or parallel to $\mathbf{a}$, we see that either both projections are zero (which happens if and only if $\mathbf{a}$ and $\mathbf{b}$ are orthogonal), or they are parallel. Moreover, if they are parallel, then in order to be equal they have to have the same length and direction. So $\mathrm{proj}_{\mathbf{b}}\mathbf{a} = \mathrm{proj}_{\mathbf{a}}\mathbf{b}$ if and only if $\mathbf{a}\cdot\mathbf{b} = 0$ or $\mathbf{a} = \mathbf{b}$.

4. (1.3.25 in *Colley*) Use vectors to show that the diagonals of a parallelogram have the same length if and only if the parallelogram is a rectangle. (Hint: let $\mathbf{a}$ and $\mathbf{b}$ be vectors along two sides of the parallelogram, and express vectors running along the diagonals in terms of $\mathbf{a}$ and $\mathbf{b}$.)

Solution. The diagonal vectors are $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$. The lengths of these vectors are equal if and only if $\|\mathbf{a}+\mathbf{b}\|^2 = \|\mathbf{a}-\mathbf{b}\|^2$. The left-hand side simplifies to $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = \mathbf{a}\cdot\mathbf{a}+2\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b}$. Simplifying the right-hand side similarly and canceling terms, we are left with $\mathbf{a}\cdot\mathbf{b} = 0$, which says that the two sides of the parallelogram are perpendicular.

5. (1.3.23 in *Colley*) Let A, B, and C denote the vertices of a triangle. Let $0 < r < 1$. If P_1 is the point on $\overline{AB}$ located r times the distance from A to B and P_2 is the point on $\overline{AC}$ located r times the distance from A to C, use vectors to show that $\overline{P_1P_2}$ is parallel to $\overline{BC}$ and has r times the length of $\overline{BC}$.

Solution. This is an application of the distributive property of scalar multiplication across vector addition. Let $\mathbf{u}$ be the vector from A to B, and let $\mathbf{v}$ be the vector from A to C. Then the vector from B to C is $\mathbf{u}-\mathbf{v}$, and the vector from P_1 to P_2 is $r\mathbf{u}-r\mathbf{v}$. Reverse distributing, we write this as $r(\mathbf{u}-\mathbf{v})$, which says that $\overline{P_1P_2}$ is parallel to $\overline{BC}$ and has r times the length of $\overline{BC}$.

6. (1975 USAMO) Let A, B, C, and D be four points in $\mathbb{R}^3$. Use vectors to show that

$$AB^2 + BC^2 + CD^2 + DA^2 \geq AC^2 + BD^2.$$

(This generalizes the fact that the sum of the squares of the sides of a quadrilateral is at least the sum of the squares of its diagonals.) Make a statement about when equality holds.

Solution. Let $\mathbf{u}$ be the vector from A to B, let $\mathbf{v}$ be the vector from A to D, and Let $\mathbf{w}$ be the vector from B to C. Then we have been asked to prove

$$\mathbf{u}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{v}+\mathbf{w}\cdot\mathbf{w}+(\mathbf{u}+\mathbf{w}-\mathbf{v})\cdot(\mathbf{u}+\mathbf{w}-\mathbf{v}) \geq (\mathbf{u}+\mathbf{w})\cdot(\mathbf{u}+\mathbf{w})+(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})$$

Distributing and simplifying, we see that this inequality is equivalent to $\|\mathbf{w}-\mathbf{v}\|^2 \geq 0$. This holds with equality if and only if $\mathbf{w}=\mathbf{v}$, i.e., if and only if the four points form a parallelogram.