

18.022 Recitation Handout
10 December 2014

1. Consider a rotating cylindrical container of radius R with a vertical axis, filled with liquid mercury up to height h . In this problem, we'll use calculus of variations to find the shape of the surface of the liquid.

(a) Assume that the surface of the liquid is described by the equation $z = f(r)$ in cylindrical coordinates, where f is some function. Sketch a graph of f based on physical intuition. Indicate the locations of R and h on your axes.

(b) Calculate the potential energy V of the liquid, assuming the density of the liquid is σ and using g to denote the gravitational constant. Express your answer as an integral in involving f .

(c) Calculate the kinetic energy T of the liquid, assuming the angular velocity of the cylinder is ω . Express your answer as an integral in involving f .

(d) Calculate the volume of the liquid, as an integral in involving f .

(e) Hamilton's principle says that the shape f will minimize $T - V$. Also, a generalization of Lagrange multipliers along with the Euler-Lagrange equations implies the following. The extrema $f(r)$ of $\int_{r_1}^{r_2} G(r, f, f') dr$ subject to the constraint $\int_{r_1}^{r_2} H(r, f, f') dr = C$ for some constant C are solutions of the system

$$\frac{d}{dr} \left(\frac{\partial(G - \lambda H)}{\partial f'} \right) = \frac{\partial(G - \lambda H)}{\partial f}$$
$$\int_{r_1}^{r_2} H(r, f, f') dr = C.$$

Note that f' is regarded as an independent variable. Put together the answers from the previous three parts to find f .

(f) Explain why your answer shows that the shape of the surface is independent of the density of the liquid.