18.022 Recitation Handout
8 December 2014

Let $D \subset \mathbb{R}^2$ be the region enclosed by the curve $r = g(\theta)$, for some C^1, non-negative $g\colon \mathbb{R} \to \mathbb{R}$ such that $g(x + 2\pi) = g(x)$ for all $x \in \mathbb{R}$.

1. Calculate the length of ∂D, the boundary of D. Express your answer as an integral involving g and its first derivative.

2. Let

$$(x(\theta), y(\theta)) = (g(\theta)\cos\theta, g(\theta)\sin\theta)$$

be a parametrization of ∂D. Calculate the length of ∂D again, but this time express the answer as an integral involving the derivatives of x and y.

3. Calculate the length of ∂D for the case that $g(\theta) = 1 - \cos\theta$.

4. In the remainder of this problem we will prove an important theorem, called the *isoperimetric inequality* (this proof is due to E. Schmidt, from 1938): it states that the length of the boundary of any shape on the plane is at least equal to the square root of 4π times its area.

Let C be the unit circle. Explain why

$$\text{area}(D) + \pi = \oint_{\partial D} (0, x) \cdot d\mathbf{s} + \oint_C (-y, 0) \cdot d\mathbf{s}. \tag{1}$$

5. Assume henceforth that $g(x) \leq 1$ and $g(0) = g(\pi) = 1$. Show that

$$(x(\theta), w(\theta)) = \begin{cases} (g(\theta)\cos\theta, \sqrt{1 - g(\theta)^2\cos^2\theta} & 0 \leq \theta \leq \pi \\ (g(\theta)\cos\theta, -\sqrt{1 - g(\theta)^2\cos^2\theta} & \pi \leq \theta \leq 2\pi \end{cases} \tag{2}$$

is a parametrization of a unit circle.

6. Let $(x(\theta), w(\theta))$ be the parametrization of the unit circle C from (2). Again let

$$(x(\theta), y(\theta)) = (g(\theta)\cos\theta, g(\theta)\sin\theta)$$

be a parametrization of ∂D. Using (1), show that

$$\text{area}(D) + \pi = \oint_0^{2\pi} (x(\theta), -w(\theta)) \cdot (y'(\theta), x'(\theta))\, d\theta.$$

7. Explain why it follows from the previous question that

$$\operatorname{area}(D) + \pi \leq \oint_0^{2\pi} \sqrt{(x(\theta)^2 + w(\theta)^2) \cdot (x'(\theta)^2 + y'(\theta)^2)}\, \mathrm{d}\theta.$$

8. Explain why

$$\operatorname{area}(D) + \pi \leq \operatorname{length}(\partial D).$$

9. Recall the AMGM inequality: for $a, b > 0$ it holds that $\sqrt{ab} \leq (a+b)/2$. Use this to show that

$$\sqrt{4\pi \cdot \operatorname{area}(D)} \leq \operatorname{length}(\partial D).$$

For which shape are these two quantities equal?