18.022 Recitation Handout (with solutions)
22 October 2014

1. Verify that divergence, curl, and gradient are linear operators.

Solution. For divergence, we want to show that for all vector fields $\mathbf{F}$ and $\mathbf{G}$ and scalars α and β, we have

$$\nabla \cdot (\alpha \mathbf{F} + \beta \mathbf{G}) = \alpha \nabla \cdot \mathbf{F} + \beta \nabla \cdot \mathbf{G}.$$

The left-hand side is

$$\frac{\partial}{\partial x}(\alpha F_1 + \beta G_1) + \frac{\partial}{\partial y}(\alpha F_2 + \beta G_2) + \frac{\partial}{\partial z}(\alpha F_3 + \beta G_3) = \alpha \frac{\partial F_1}{\partial x} + \beta \frac{\partial G_1}{\partial x} + \alpha \frac{\partial F_2}{\partial y} + \beta \frac{\partial G_2}{\partial y} + \alpha \frac{\partial F_3}{\partial z} + \beta \frac{\partial G_3}{\partial z},$$

which equals the right-hand side. Calculations for gradient and curl are similar.

2. Let $\mathbf{F}(x, y, z) = (3x^2 + \frac{1}{2}y^2 + e^z, xy + z, f(x, y, z))$. Find all f such that $\mathbf{F}$ is curl-free.

Solution. The third component of $\nabla \times \mathbf{F}$ is $y - (1/2)(2y) = 0$, as desired. For the first component to be zero, we must have $f_y = 1$, and for the second component to be zero we must have $f_x = e^z$. Integrating these two equations tells us that $f(x, y, z) = y + C_1(x, z)$ and $f(x, y, z) = xe^z + C_2(y, z)$ for functions C_1 and C_2 which do not depend on y or x respectively. Putting these two together, we see that $f(x, y, z) = xe^z + y + C(z)$ for any differentiable function C.

3. Confirm that for a vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$, we have

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F},$$

where $\nabla^2 \mathbf{F}$ is defined to mean "take the Laplacian of each component of $\mathbf{F}$." Is it possible to derive this identity from $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$?

Solution. [Omitted]

4. Let $\mathbf{F}$ be a C^2 vector field on $\mathbb{R}^3$. Show that $\nabla \times \mathbf{F}$ is incompressible.

Solution. We calculate

$$\nabla \cdot (\nabla \times \mathbf{F}) = \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} = 0,$$

since the mixed partials don't depend on the order of differentiation, as $\mathbf{F}$ is C^2.