18.022 Recitation Handout
15 September 2014

1. Find the cosine of the angle between the vectors $(-4, 2, 6, 5)$ and $(3, 1, 0, -7)$.

2. Is the cross-product associative? Is the dot product associative? Prove or give a counterexample for each.

3. Find an equation for the plane that contains the lines

$$\begin{cases} x(t) = 5 + t \\ y(t) = 1 - t \\ z(t) = 4 - 3t. \end{cases}$$

and

$$\begin{cases} x(t) = 5 - 4t \\ y(t) = 1 + t \\ z(t) = 4 - 3t. \end{cases}$$

4. Find the distance from the origin to the plane $x + y + z = 1$.

5. Give a geometric description of what each of the following matrices does to a vector.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

6. What matrices have the following geometric descriptions?

(a) reflect across the x-axis

(b) reverse the direction of the vector and double its length

(c) halve the length of the vector (while preserving the direction)

(d) rotate a vector 90 degrees counterclockwise

(e) project a vector in $\mathbb{R}^3$ onto the x-y plane

7. The volume of a parallelepiped is the product of the area of its base and its height. Consider the parallelepiped spanned by **a**, **b**, and **c**. You may suppose for simplicity that **b**, **c**, and **a** form a right-handed triple of vectors, which means that a right-handed screw rotated an angle less than 180° from **b** to **c** advances in the direction of **a**.

(a) Let us think of the parallelogram spanned by **b** and **c** as the base of the parallelepiped. What is the (signed) area of this parallelogram?

(b) What is the height of the parallelogram, in terms of **a** and the unit vector pointing in the direction of $\mathbf{b} \times \mathbf{c}$?

(c) Put together parts (a) and (b) to derive the triple scalar product formula for the volume of a parallelepiped.

8. (Fun/Challenge problem) It is possible to prove the vector triple product formula

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

in a tedious way using coordinates. This problem outlines a more conceptual proof, taken from a note written by William C. Schulz.

(a) Use the paralellepiped interpretation of the triple scalar product to show that

$$\mathbf{b} \cdot (\mathbf{c} \times \mathbf{n}) = \mathbf{c} \cdot (\mathbf{n} \times \mathbf{b}) = \mathbf{n} \cdot (\mathbf{b} \times \mathbf{c})$$

(b) Use the right-hand rule to observe that if $\mathbf{c}$ is perpendicular to $\mathbf{n}$, then

$$\mathbf{n} \times (\mathbf{c} \times \mathbf{n}) = |\mathbf{n}|^2 \mathbf{c}.$$

(c) Show that it suffices to consider the case where $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ form a basis for $\mathbb{R}^3$.

(d) Write $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ as a linear combination of $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$, so that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \kappa \mathbf{a} + \lambda \mathbf{b} + \mu \mathbf{c}, \tag{1}$$

(e) The easiest coefficient to determine is κ. What is it?

(f) To find λ, dot both sides of (1) with $\mathbf{c} \times \mathbf{n}$, where $\mathbf{n} := \mathbf{b} \times \mathbf{c}$.

(g) To find μ, dot both sides of (1) with $\mathbf{b} \times \mathbf{n}$, where $\mathbf{n} := \mathbf{b} \times \mathbf{c}$.