18.022 Recitation Quiz (with solutions)
01 December 2014

1. Let S be the surface defined by

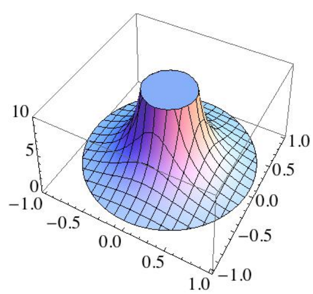
$$z = \frac{1}{x^2 + y^2} \text{ for } z \geq 1.$$

(a) Sketch the graph of this surface.

(b) Determine whether the volume of the region bounded by S and the plane $z = 1$ is finite or infinite.

(c) Determine whether the surface area of S is finite or infinite.

Solution. (a) See the graph below (cut off at $z = 10$).



(b) The volume is given by $\int_0^1 \int_0^2 \pi(r^{-2} - 1) r \, d\theta \, dr = \int_0^1 \int_0^2 \pi(r^{-1} - r) \, d\theta \, dr$. This is infinite since $\int_0^1 \frac{dr}{r}$ is infinite and $\int_0^1 r \, dr$ is finite. (c) The surface area of S is given by

$$\begin{aligned}\int_0^1 \int_0^{2\pi} \sqrt{1 + f_x^2 + f_y^2} \, r \, d\theta \, dr &= \int_0^1 \int_0^{2\pi} \sqrt{1 + x^2/(x^2+y^2)^4 + y^2/(x^2+y^2)^4} \, r \, d\theta \, dr \\ &= \int_0^1 \int_0^{2\pi} \sqrt{1 + r^{-6}} \, r \, d\theta \, dr \\ &= 2\pi \int_0^1 \sqrt{r^2 + r^{-4}} \, dr.\end{aligned}$$

This integral is infinite because the second term dominates, and $\int_0^1 \frac{dr}{r^2} = +\infty$. To prove this rigorously, we can drop the first term:

$$2\pi \int_0^1 \sqrt{r^2 + r^{-4}} \, dr \geq 2\pi \int_0^1 \sqrt{r^{-4}} \, dr = +\infty. \qquad \square$$