

18.022 Recitation Quiz (with solutions)  
20 October 2014

1. Suppose that  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  and  $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  are differentiable. Find the total derivative  $D(f \circ g)$ . (Note: you may write  $f$  as  $(f_1, f_2)$ , where  $f_i : \mathbb{R}^2 \rightarrow \mathbb{R}$  for  $i \in \{1, 2\}$ , and similarly for  $g$ .)

*Solution.* Let's use the variables  $s$  and  $t$  for the arguments of  $g$ , and let's use the variables  $x$  and  $y$  for the arguments of  $f$ . The composition  $f \circ g$  is given by

$$(f \circ g)(s, t) = (f_1(g_1(s, t), g_2(s, t)), f_2(g_1(s, t), g_2(s, t))).$$

By the chain rule, the partial derivative of  $f_1(g_1(s, t), g_2(s, t))$  with respect to  $s$  is

$$\frac{\partial f_1}{\partial x} \frac{\partial g_1}{\partial s} + \frac{\partial f_1}{\partial y} \frac{\partial g_2}{\partial s}.$$

Doing similar calculations for the other terms, we end up with

$$(D(f \circ g))(s, t) = \begin{pmatrix} \frac{\partial f_1}{\partial x} \frac{\partial g_1}{\partial s} + \frac{\partial f_1}{\partial y} \frac{\partial g_2}{\partial s} & \frac{\partial f_1}{\partial x} \frac{\partial g_1}{\partial t} + \frac{\partial f_1}{\partial y} \frac{\partial g_2}{\partial t} \\ \frac{\partial f_2}{\partial x} \frac{\partial g_1}{\partial s} + \frac{\partial f_2}{\partial y} \frac{\partial g_2}{\partial s} & \frac{\partial f_2}{\partial x} \frac{\partial g_1}{\partial t} + \frac{\partial f_2}{\partial y} \frac{\partial g_2}{\partial t} \end{pmatrix}.$$