18.022 Recitation Quiz (with solutions)
15 October 2014

1. Consider the function $f(x) = \sqrt{1-x^2}$ over the interval $[0,1]$. Write down a definite integral whose value is equal to the arclength of the graph of f.

Solution. We calculate an arclength of

$$\int_0^1 \sqrt{1+f'(x)^2}\,dx = \int_0^1 \sqrt{1+\left(\frac{-2x}{2\sqrt{1-x^2}}\right)^2}\,dx = \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx.$$

Remark: Since this arclength is $\pi/2$ by the definition of π, this exercise proves that $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \pi/2$. Moreover, it can be generalized to give a way of calculating the indefinite integral of $1/\sqrt{1-x^2}$.

2. Consider the function $\mathbf{F} : \mathbb{R}^4 \to \mathbb{R}^2$ defined by $\mathbf{F}(w,x,y,z) = (2/w^2 - y, 3x + \cos z)$.

(a) Find $D\mathbf{F}$.

Solution. The total derivative is the matrix of partial derivatives:

$$\begin{pmatrix} -2/w^3 & 0 & -1 & 0 \\ 0 & 3 & 0 & -\sin z \end{pmatrix}$$

(b) Show that there exists an open set $U \subset \mathbb{R}^2$ containing $(1,2)$ and a function $\mathbf{f} : U \to \mathbb{R}^2$ such that for all $x \in U$, the equations $\mathbf{F}(w,x,y,z) = \mathbf{F}(1,2,3,\pi/2)$ have a unique solution $(y,z) = \mathbf{f}(w,x)$. Show that $\mathbf{f}$ is C^1.

Solution. The implicit function theorem ensures that we can solve (abstractly) for (y,z) in terms of x and w if the matrix of partial derivatives corresponding to the y and z columns has nonvanishing determinant. In this case, that means

$$\det\begin{pmatrix} -1 & 0 \\ 0 & -\sin z \end{pmatrix} = \det\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = 1 \neq 0,$$

so the implicit function theorem does apply. It ensures the existence of such an $\mathbf{f}$ and the fact that $\mathbf{f}$ is C^1.