18.022 Practice Final Exam (with solutions)
13 December 2014

1. Calculate the flux of the vector field $\mathbf{F} = (3x, 2y, 0)$ through the unit sphere in $\mathbb{R}^3$

Solution. The divergence of **F** is the constant function 5, so the divergence theorem tells us that the flux equals 5 times the volume of the sphere, which is $\boxed{20\pi/3}$.

2. Compute both sides of the equation in the statement of the divergence theorem for the vector field $\mathbf{F} = (x, y, z)/(x^2 + y^2 + z^2)$ and the unit sphere in $\mathbb{R}^3$. Are the hypotheses of the divergence theorem satisfied in this case? Why or why not?

Solution. The divergence of F is $1/(x^2 + y^2 + z^2)$, and its integral over the unit sphere is $\boxed{4\pi}$. The flux integral is given by

$$\int_{\text{unit sphere}} \frac{(x, y, z)}{x^2 + y^2 + z^2} \cdot \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{1/2}} \, dS = \boxed{4\pi}.$$

So the conclusion of the divergence theorem happens to be satisfied. However, the divergence theorem does not apply in this case, because **F** is not differentiable (or even defined) at the origin.

3. (a) Consider a particle moving in $\mathbb{R}^3$ so that its location at time t is given by $(\sin(t^2), \cos(t^2), t)$, where t ranges over the interval $[0, \sqrt{8\pi}]$. Find the speed of the particle at time t as well as its maximum speed.

Solution. We have $\mathbf{r}'(t) = (2t\sin(t^2), -2t\cos(t^2), 1)$, so

$$|\mathbf{r}'(t)| = \sqrt{4t^2 + 1},$$

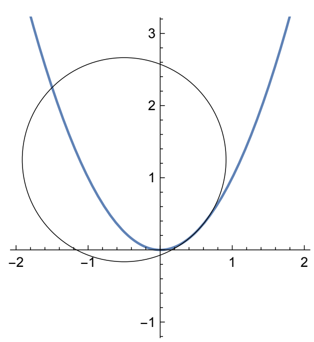
which has a maximum value of $\boxed{\sqrt{32\pi + 1}}$.

(b) How far did the particle go? You may leave your answer as an unevaluated definite integral.

Solution. To get distance from speed, we integrate:

$$\text{total distance} = \int_0^{\sqrt{8\pi}} \sqrt{4t^2 + 1} \, dt.$$

4. Calculate the area of the osculating circle at the point $(0.5, 0.25)$ for the parabola $y = x^2$, as shown below. Hint: the radius of the osculating circle at a point is equal to the reciprocal of the curvature at that point.



Solution. Parametrizing the parabola as (t, t^2), we find that the curvature is given by $2/(1+4t^2)^{3/2}$. So the area of the osculating circle is

$$\pi\left(\frac{(1+4t^2)^{3/2}}{2}\right)^2 = \boxed{2\pi}.$$

5. The AMGM inequality states that for all $x, y \geq 0$, we have

$$\sqrt{xy} \leq \frac{x+y}{2}.$$

Use the method of Lagrange multipliers to prove this inequality by minimizing $(x+y)/2$ subject to the constraint $\sqrt{xy} = c$, where c is a constant.

Solution. Letting $f(x, y) = (x+y)/2$ and $g(x, y) = \sqrt{xy}$, we get from the Lagrange multiplier equation $\nabla f = \lambda \nabla g$,

$$1/2 = \lambda\sqrt{y/x}/2, \quad 1/2 = \lambda\sqrt{x/y}/2, \quad \sqrt{xy} = c.$$

This system is solved only when $x = y = c$, so the only critical point of f subject to the constraint $g(x, y) = c$ occurs when $f(x, y) = (c+c)/2 = c$. To show that this is minimum of f, note that the arithmetic mean is clearly larger than the geometric mean when either x or y is large. To be more precise, we could argue that f does not achieve its minimum when either x or y is greater than some large constant C, and we can use the extreme value theorem to show that it has to achieve some minimum on the set of pairs (x, y) for which x and y are both less than or equal to C and $\sqrt{xy} = c$.

6. Let A and B be two points on the surface of a sphere which are as far apart as possible, and let C be the point $1/4$ of the way from A to B (on the line segment from A to B). Cut the sphere into two pieces along a plane passing through C and perpendicular to AB. What is the ratio of the volume of the larger piece to the smaller?

Solution. We may assume without loss of generality that the sphere has unit radius. The volume of the small piece is given in spherical coordinates by

$$\int_0^{2\pi}\int_0^{\pi/3}\int_{\frac{1}{2}\sec\phi}^{1} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 5\pi/24.$$

The volume of the larger piece is $\frac{4\pi}{3} - \frac{5\pi}{24} = 9\pi/8$, so the ratio is $\boxed{5/27}$.

7. Use the Euler-Lagrange equations to show that the shortest distance between two points in $\mathbb{R}^2$ is a straight line.

Solution. See this page for a solution.

8. Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is differentiable and that $x_0 \in \mathbb{R}^3$. Suppose that $\nabla f(x_0) = 2.5$ and

$$Hf(x_0) = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ 3 & 1 & 0 \end{pmatrix},$$

where Hf denotes the Hessian of f. Does f have a local maximum or minimum at x_0? Answer the same question assuming instead that $\nabla f(x_0) = 0$ and $Hf(x_0)$ is the same as above.

Solution. If the gradient of f at x_0 is 2.5, then in particular it is nonzero. Thus f has neither a local maximum nor a local minimum at x_0. If the gradient is 0, then f has a local max, local min, or saddle point according to whether the Hessian is positive definite, negative definite, or neither. In this case, if we calculate $(1,0,0)H(1,0,0)^T$, we get 1, and if we calculate $(1,0,0)H(1,0,0)^T$, we get -1. Thus it is neither the case that $vHv^T > 0$ for all $v \in \mathbb{R}^3$ nor that $vHv^T < 0$ for all $v \in \mathbb{R}^3$. Therefore, the Hessian is neither positive definite nor negative definite, and we conclude that f has a saddle point at x_0.