18.022 Practice Final Exam
13 December 2014

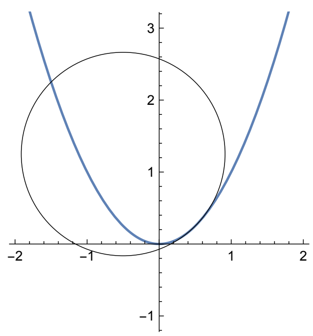
1. Calculate the flux of the vector field $\mathbf{F} = (3x, 2y, 0)$ through the unit sphere in $\mathbb{R}^3$

2. Compute both sides of the equation in the statement of the divergence theorem for the vector field $\mathbf{F} = (x, y, z)/(x^2 + y^2 + z^2)$ and the unit sphere in $\mathbb{R}^3$. Are the hypotheses of the divergence theorem satisfied in this case? Why or why not?

3. (a) Consider a particle moving in $\mathbb{R}^3$ so that its location at time $t$ is given by $(\sin(t^2), \cos(t^2), t)$, where $t$ ranges over the interval $[0, \sqrt{8\pi}]$. Find the speed of the particle at time $t$ as well as its maximum speed.

(b) How far did the particle go? You may leave your answer as an unevaluated definite integral.

4. Calculate the area of the osculating circle at the point $(0.5, 0.25)$ for the parabola $y = x^2$, as shown below. Hint: the radius of the osculating circle at a point is equal to the reciprocal of the curvature at that point.




5. The AMGM inequality states that for all $x, y \geq 0$, we have

$$\sqrt{xy} \leq \frac{x+y}{2}.$$

Use the method of Lagrange multipliers to prove this inequality by minimizing $(x+y)/2$ subject to the constraint $\sqrt{xy} = c$, where $c$ is a constant.

6. Let $A$ and $B$ be two points on the surface of a sphere which are as far apart as possible, and let $C$ be the point $1/4$ of the way from $A$ to $B$ (on the line segment from $A$ to $B$). Cut the sphere into two pieces along a plane passing through $C$ and perpendicular to $AB$. What is the ratio of the volume of the larger piece to the smaller?

7. Use the Euler-Lagrange equations to show that the shortest distance between two points in $\mathbb{R}^2$ is a straight line.

8. Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is differentiable and that $x_0 \in \mathbb{R}^3$. Suppose that $\nabla f(x_0) = 2.5$ and

$$Hf(x_0) = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ 3 & 1 & 0 \end{pmatrix},$$

where $Hf$ denotes the Hessian of $f$. Does $f$ have a local maximum or minimum at $x_0$? Answer the same question assuming instead that $\nabla f(x_0) = 0$ and $Hf(x_0)$ is the same as above.