BROWN UNIVERSITY
Math 1610 – Probability
Exam 1
Instructor: Samuel S. Watson
Due: 15 October 2015

Key

1. (12 points) Show that if (Ω, P) is a probability space and $E \subset F \subset \Omega$ are events, then

$$P(F \setminus E) = P(F) - P(E).$$

You may assume only the axioms of a probability space: $P(E) \geq 0$ for all events $E \subset \Omega$, $P(\Omega) = 1$, and $P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$ for all pairwise disjoint sequences of events $(E_i)_{i=1}^{\infty}$.

$P(F \cup E) = P(F) \cup P(E)$ from countable additivity by taking $E_3 = E_4 = \cdots = \emptyset$.

Then

$$P(E) + P(F \setminus E) = P(E \cup (F \setminus E)) = P(F)$$

by additivity, so

$$P(F) - P(E) = P(F \setminus E),$$

as desired.

2. (12 points) Research shows that, for the average coin flipper, the result of a coin flip has an approximately 51% chance to be the same side which was facing up when the coin was flipped.

A coin is flipped 4 times, with the first flip being fair and the remaining 3 flips each having 51% probability of being the same as the result of the previous flip. Write down an arithmetic expression for the probability of that at least 3 of the 4 flips are heads. You do not have to evaluate the expression.

$P(3 \text{ or } 4 \text{ heads})$

$$= P(HHHT) + P(HHTH) + P(HHHH) + P(THHH) + P(HTHH)$$

$$= \left(\tfrac{1}{2}\right)(0.51)^2(0.49) + \left(\tfrac{1}{2}\right)(0.51)(0.49)^2 + \left(\tfrac{1}{2}\right)(0.51)^3 + \left(\tfrac{1}{2}\right)(0.51)^2(0.49) + \left(\tfrac{1}{2}\right)(0.51)(0.49)^2$$

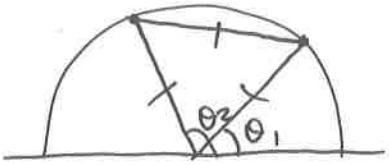
$$= \tfrac{1}{2}(0.51)^3 + 2\cdot\tfrac{1}{2}(0.51)^2(0.49) + 2\cdot\tfrac{1}{2}(0.51)(0.49)^2$$

3. A random chord of the unit circle is selected by choosing one endpoint uniformly at random from the portion of the circle in the first quadrant and and choosing the second endpoint uniformly at random from the portion of the circle in the second quadrant.

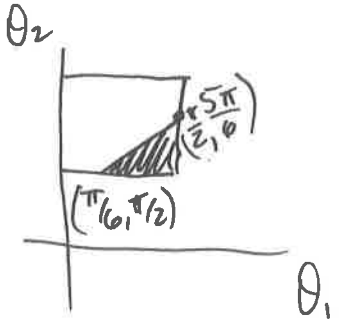
(a) (2 points) Carefully state a probability space (Ω, P) to model this random experiment.

$\Omega = [0, \frac{\pi}{2}] \times [\frac{\pi}{2}, \pi]$, P = uniform (area) measure.

(b) (8 points) Find the probability that the length of the random chord exceeds one unit.



length ≥ 1 iff $\theta_2 - \theta_1 \geq \pi/3$, so the prob. is one minus the area of the triangle shown (the intersection of Ω and $\theta_2 \geq \theta_1 + \pi/3$) divided by $(\frac{\pi}{2})^2$:



$$\text{prob} = 1 - \frac{\frac{1}{2}\left(\frac{\pi}{3}\right)^2}{\left(\frac{\pi}{2}\right)^2}$$

$$= 1 - \frac{\pi^2}{2\ 9} \cdot \frac{4}{\pi^2} = \boxed{1 - \frac{2}{9}} = \boxed{\frac{7}{9}}$$

4. Consider the function $F : \mathbb{R} \to [0,1]$ defined by

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } 0 < x < 1/2 \\ 1 & \text{if } 1/2 \leq x. \end{cases}$$

If X is a random variable whose cdf is equal to F, find each of the the following probabilities:

(a) (4 points) $P(1/4 \leq X \leq 1/2)$

$\lim_{x \uparrow \frac{1}{4}} F(x)$

$$P(X \in [1/4, 1/2]) = F(\tfrac{1}{2}) - F(\tfrac{1}{4}-)$$

$$= 1 - \frac{1}{4} = \boxed{\frac{3}{4}}$$

(b) (4 points) $P(X = 1/2)$

$$P(X = 1/2) = F(\tfrac{1}{2}) - F(\tfrac{1}{2}-)$$

$$= \boxed{\frac{1}{2}}$$

(c) (3 points) $P(X = 1/3)$

$$P(X = 1/3) = F(\tfrac{1}{3}) - F(\tfrac{1}{3}-) = \boxed{0}.$$

(d) (3 points) Comment on why X does **not** have a probability density function.

F is not differentiable, so X has no pdf.

5. (12 points) Use Stirling's formula to find a simple expression which is asymptotic to $\binom{4N}{N}$ as $N \to \infty$.

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$\frac{(4N)!}{N!\,(3N)!} \sim \frac{\left(\frac{4N}{e}\right)^{4N}\sqrt{2\pi(4N)}}{\left(\frac{N}{e}\right)^{N}\sqrt{2\pi N}\,\left(\frac{3N}{e}\right)^{3N}\sqrt{2\pi(3N)}}$$

$$= \frac{4^{4N}\,\cancel{N^{4N}}\,\cancel{e^{-4N}}\,\sqrt{8\pi N}}{3^{3N}\,\cancel{N^{4N}}\,\cancel{e^{-4N}}\sqrt{2\pi N}\,\sqrt{2\pi(3N)}}$$

$$= \left(\frac{128}{27}\right)^N \sqrt{\frac{8\pi N}{2\pi N \cdot 6\pi N}}$$

$$= \left(\frac{128}{27}\right)^N \sqrt{\frac{2}{3\pi N}}$$

6. (12 points) Consider ten independent, fair coin flips. Write an expression for the conditional probability that $\underbrace{\text{the first three flips are heads}}_{A}$ given that $\underbrace{\text{there are exactly 7 heads total}}_{B}$.

$$P(A \mid B) = \frac{P(A \& B)}{P(B)}$$

$$= \frac{P(A \text{ and four of last 7 are heads})}{P(\text{exactly 7 heads})}$$

$$= \frac{\left(\frac{1}{2}\right)^3 \binom{7}{4} \left(\frac{1}{2}\right)^7}{\binom{10}{7}\left(\frac{1}{2}\right)^{10}}$$

$$\frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$$

$$= \frac{\binom{7}{4}}{\binom{10}{7}}$$

$$\frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$$

$$= \frac{35}{120} = \boxed{\frac{7}{24}}$$

7. Suppose the joint law of X and Y is given by the pdf

$$f(x,y) = \frac{3}{2}(x^2 + y^2)$$

on the square $[0,1]^2$.

(a) (4 points) Are X and Y independent? Explain.

No, because f cannot be written as a product of a function of x and a function of y.

(b) (10 points) Find the conditional probability that $X \geq 3/4$ given $Y \geq 3/4$.

$$P(X \geq 3/4 \mid Y \geq 3/4) = \frac{P(X \geq 3/4 \text{ and } Y \geq 3/4)}{P(Y \geq 3/4)}$$

$$= \frac{\int_{3/4}^1 \int_{3/4}^1 \frac{3}{2}(x^2+y^2)\,dx\,dy}{\int_0^1 \int_{3/4}^1 \frac{3}{2}(x^2+y^2)\,dy\,dx}$$

$$1 - \left(\frac{3}{4}\right)^3 = 1 - \frac{27}{64} = \frac{37}{64}$$

$$= \frac{\int_{3/4}^1 \frac{1}{2}x^3 + \frac{3}{2}xy^2 \Big|_{3/4}^1 dy}{\int_0^1 \frac{1}{2}y^3 + \frac{3}{2}yx^2 \Big|_{3/4}^1 dx}$$

$$= \frac{\int_{3/4}^1 \frac{1}{2}\cdot\frac{37}{64} + \frac{3}{2}\cdot\frac{1}{4}\cdot y^2\,dy}{\int_0^1 \frac{1}{2}\cdot\frac{37}{64} + \frac{3}{2}\cdot\frac{1}{4}\cdot x^2\,dx}$$

$$= \frac{\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{37}{64} + \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{37}{64}}{\frac{1}{2}\cdot\frac{37}{64} + \frac{1}{2}\cdot\frac{1}{4}\cdot 1}$$

$$\frac{\overset{16}{37}}{53}$$

$$= \frac{1}{4}\cdot\frac{37}{64}\cdot\frac{128}{53} = \boxed{\frac{37}{106}}$$

8. (12 points) Two sealed envelopes contain fixed but unknown real numbers X and Y with $X \neq Y$ (if you want, you may think of these numbers as having been selected by an adversary who wishes to make your task as difficult as possible). You have no other information about X and Y. You select one of the envelopes uniformly at random and discover the value of the number contained therein. At that point you must say whether the discovered number or the number still sealed is larger. You have access to a random number generator, and your goal is to be correct with probability strictly greater than 1/2.

Perhaps surprisingly, this task is possible.

(a) Show that there exists a random variable Z such that $P(Z \in [a, b]) > 0$ for all $-\infty < a < b < \infty$ (note: there are many such random variables).

Let $Z \sim N(0,1)$. Then $P(Z \in [a,b]) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt > 0$.

(b) Devise a strategy using the random variable Z from part (a) and show that it succeeds with probability greater than 1/2.

~~Idea~~ Swith to the envelope if ~~Z > X~~ Z is greater than the discovered value and stay if Z is less. Then you're guaranteed to get it right if ~~Z~~ Z is between X and Y, and otherwise your chances are 50-50. Since Z is between X and Y with probability greater than zero, you succeed with probability greater than 1/2.