Homework #8
Solutions

1. The Geometric distribution is slowest.

The fair die distribution is maybe a little faster than the Poisson distribution.

2. The sum S_{25} has mean 250 and variance $25 \cdot \frac{100}{3}$, so the density is roughly

$$\frac{1}{\sqrt{25 \cdot \frac{100}{3}}\sqrt{2\pi}} e^{-(x-250)^2/2(25\cdot 100/3)}.$$

S^*_{250} of course is modeled by $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, the $N(0,1)$ density,

A_{25} has mean 10 and variance $\frac{100}{3} \cdot \frac{1}{250}$,

so its density is roughly

$$\frac{1}{\sqrt{\frac{100}{3} \cdot \frac{1}{250}}\sqrt{2\pi}} e^{-(t-10)^2/2 \cdot 100/3 \cdot \frac{1}{250}}.$$

3. Chebyshev: $\frac{10}{n} \le 0.05$, $n \geq 200$.

CLT: $2\sqrt{10/n} \approx 1$, $n \approx 40$.

Chebyshev: $\sigma^2/10 \approx 0.05$, $\sigma^2 = 0.5$

CLT: $2\sqrt{\sigma^2/10} \approx 1$, $\sigma^2 = 2.5$.

4(a) mean $= 0$, and variance is

$$\mathrm{Var}\left(\frac{1}{n}\sum_{k=1}^{n} V_k\right) = \frac{1}{n^2}\sum \mathrm{Var}\, V_k = \frac{1}{n^2}\cdot n = \frac{1}{n} = 10^{-20}.$$

(b) The pba the average velocity is 10^{-9} cm/sec is

$$P(A > 10^{-9}) = P\left(\frac{A}{10^{-10}} > \frac{10^{-9}}{10^{-10}}\right)$$

$$\approx P(Z > 10)$$

$$= 7.6 \times 10^{-24}.$$

5. (a) $Ee^{tX} = \frac{1}{n}\left(e^{tn} + e^{t(n+1)} + \cdots + e^{t(n+k)}\right)$

$$= \frac{1}{n}\left(\frac{e^{t(n+k+1)} - e^{tn}}{e^t - 1}\right)$$

(b) $Ee^{tX} = e^{3t}$

(c) $Ee^{tX} = \frac{1}{2}(e^t + e^{-t}) = \cosh(t)$. go back one step

(a) mean $= \left.\frac{d}{dt}\right|_{t=0} \frac{1}{n}\left(\frac{e^{t(n+k+1)} - e^{tn}}{e^t - 1}\right) = \frac{1}{n}\left(n + (n+1) + \cdots + (n+k)\right)$

$$= k + 1 + \frac{k(k+1)}{2n}$$

~~variance~~

Second moment $= \frac{1}{n}\left(n^2+(n+1)^2+\cdots+(n+k)^2\right)$

$$= \frac{(n+k)(n+k+1)(2n+2k+1)}{6n} - \frac{(n-1)(n)(2n-1)}{6n}$$

variance = second moment minus squared mean,

$$= \frac{(n+k)(n+k+1)(2n+2k+1)}{6n} - \frac{n(n-1)(2n-1)}{6n} - \left(k+1+\frac{k(k+1)}{2n}\right)^2.$$

(b) mean $= \frac{d}{dt}\Big|_{t=0} e^{3t} = 3.$

Variance $= \left(\frac{d}{dt}\right)^2\Big|_{t=0} e^{3t} - \left(\frac{d}{dt}\Big|_{t=0} e^{3t}\right)^2 = 9-3^2=0.$

(c) mean $= \frac{d}{dt}\Big|_{t=0} \cosh(t) = \sinh(t)\Big|_{t=0} = 0.$

variance $= \left(\frac{d}{dt}\right)^2\Big|_{t=0} \cosh(t) = \cosh(0) = 1.$

6. P is characterized ~~by~~ by $P(0)$ & $P(1)$. We have

$$\mu_1 = 0\cdot P(0) + 1(P(1)) + 2\left(1-P(0)-P(1)\right)$$

$$= 2-2P(0)-P(1)$$

$$\mu_2 = 0\cdot P(0) + 1\cdot P(1) + 4\left(1-P(0)-P(1)\right)$$

$$\Rightarrow \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} P(0) \\ P(1) \end{pmatrix} = \begin{pmatrix} 2-\mu_1 \\ 4-\mu_2 \end{pmatrix}$$

since $\det\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} = 6-4 = 2 \neq 0$, this system is invertible. Thus $P(0), P(1)$ are uniquely determined by μ_1, μ_2.

7. Just let $P(n) = C|n|^{-2}$ for all $n \in \mathbb{Z} \setminus \{0\}$, where $C = \left(\frac{\pi^2}{12}\right)^{-1}$ is a normalizing constant.

8. Let $P(n) = e^{-n}$ for $n \geq 1$. Then

$$\mathbb{E}e^{tX} = \sum_{n=1}^{\infty} e^{tn} e^{-n}$$
$$= \sum_{n=1}^{\infty} e^{n(t-1)} = \begin{cases} \infty & \text{if } t > 1 \\ < \infty & \text{if } t < 1. \end{cases}$$