Homework #7

Solutions

1. $$P(\text{June passes}) = P(J \geq 30)$$

$$= P\left(J - \tfrac{3}{4}\cdot 48 \geq 30 - \tfrac{3}{4}\cdot 48\right)$$

$$= P\left(\frac{J - \frac{3}{4}\cdot 48}{\sqrt{\frac{3}{4}\cdot\frac{1}{4}}\sqrt{48}} \geq \frac{30 - \frac{3}{4}\cdot 48}{\sqrt{\frac{3}{4}\cdot\frac{1}{4}}\cdot\sqrt{48}}\right)$$

$$\approx P\left(Z \geq \frac{30 - \frac{3}{4}\cdot 48}{\sqrt{\frac{3}{4}\cdot\frac{1}{4}}\cdot\sqrt{48}}\right) = P(Z \geq -2)$$

where $Z \sim N(0,1)$, & this is

$$\frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{-2}{\sqrt{2}}\right)\right) \approx 0.977.$$

$$P(\text{April passes}) \approx P\left(Z \geq \frac{30 - \frac{1}{2}\cdot 48}{\sqrt{48\cdot\frac{1}{2}\cdot\frac{1}{2}}}\right) = P(Z \geq \sqrt{3})$$

$$\approx 0.0416,$$

using the normal approximation.

2.

	(a) Chebyshev	(b) CLT
500	$1 - \frac{\sigma^2}{500^2} = 1 - 1 = 0$	0.6826
1000	$1 - \frac{\sigma^2}{1000^2} = 1 - \frac{1}{4} = \frac{3}{4}$	0.9545
1500	$1 - \frac{\sigma^2}{1500^2} = 1 - \frac{1}{9} = \frac{8}{9}$	0.9973

3. (a) $P(A_n = 0.8) \to 0$ as $n \to \infty$ because at most one outcome involves $A_n = 0.8$ exactly, & the most likely outcome still has probability tending to 0 as $n \to \infty$.

(b) $\lim_{n\to\infty} P(0.7n < S_n < 0.9n)$

$= \lim_{n\to\infty} P(0.7 < A_n < 0.9) = 1$ by LLN.

(c) $\lim_{n\to\infty} P\left(\frac{S_n - 0.8n}{\sqrt{0.2 \cdot 0.8}\sqrt{n}} < \frac{0.8}{\sqrt{0.2 \cdot 0.8}}\right) = 0.977$ (CLT)

(d) $\lim_{n\to\infty} P(|A_n - 0.8| < 0.001) = 1$ by LLN.

4. We want $\frac{2\sqrt{p(1-p)}}{\sqrt{n}} \le 0.01$. Since $\sqrt{p(1-p)} \le 1/2$, this is ensured if $\frac{1}{\sqrt{n}} \le 0.01$, i.e., $n = 10{,}000$.

5. Let $X_i = \pm 1$, iid, for $i = 1, 2, \ldots, 100$. Then

$P\left(\frac{|S_{100}|}{\sigma} \ge \frac{10}{\sigma}\right) \approx P(|Z| \ge 10/\sigma)$ where $\sigma =$ the standard deviation of S_{100}.

$$\operatorname{Var} X_1 = E(X_1^2) - \underbrace{\cancel{(EX_1)^2}}_{\to 0}$$

$$= (-1)^2 \cdot \frac{1}{2} + (1)^2 \cdot \frac{1}{2} = 1, \text{ so } \operatorname{Var} S_{100} = 100.$$

So the answer is $P(|Z|>1) \approx 0.317$.

6. $P(S_{100} > 1000)$

$$= P\left(\frac{S_{100} - 100\cdot 10}{\sqrt{100}} > \frac{1000 - 100\cdot 10}{\sqrt{100}} \right)$$

$$\approx 1/2,$$

by symmetry.

$$P(S_{100} > 970) \approx P\left(Z > \frac{970 - 100\cdot 10}{\sqrt{100}} \right) = 0.9987$$

7. Yes; take p sufficiently close to 1. For example, if $\cancel{(1-p)^n}\ p^n > n(1-p)p^{n-1}$, i.e. $p > \frac{n}{n+1}$, then the most likely outcome is all heads, and the probability decreases with the number of tails. ~~As~~

This does not contradict the CLT, because that deals with fixed p as $n \to \infty$, & here we have $p = \frac{n}{n+1}$ depending on n.

8. (a) $E S_n^4 = \sum_{k=1}^{n} E(X_k^4) + 6 \sum_{1 \le i < j \le n} E(X_i^2 X_j^2)$,

because all the other terms have at least one X_k^1, and $E(X_k^1 X_j^p X_l^q) = 0$ for any p and q.

(b) ~~$E(S_n^4)$ $E X_k^4 \ge (E X_k^2)^2$~~

We have $E(X_i^2 X_j^2) = EX_i^2 EX_j^2 = (EX_i^2)^2 \le EX_i^4 \le K$,

So the above sum is

$$E S_n^4 \le nK + 6n(n-1)/2 \cdot K$$
$$= 3Kn^2 - 2nK \le 3Kn^2.$$

(c) So $E \sum \left(\frac{S_n}{n}\right)^4 \le \sum_n n^{-4}(3Kn^2) = 3K \sum n^{-2} < \infty$.

(d) thus $\sum_n \left(\frac{S_n}{n}\right)^4 < \infty$ with pba 1 (since otherwise $\sum \left(\frac{S_n}{n}\right)^4$ would equal ∞ with positive pba & therefore have infinite expectation.) So $\frac{S_n}{n} \to 0$ as $n \to \infty$, with probability 1.