Homework #6
Solutions

1. (a) $P(Y\in(y,y+\Delta y)) = \sum_\sigma P(X_{\sigma(1)} < \cdots < X_{\sigma(10)}$ and $X_{\sigma(3)} \in (y,y+\Delta y)) = 10!\, P(X_1 < \cdots < X_{10}$ and $X_3 \in (y,y+\Delta y))$, by symmetry.

(b) $P(X_1 < \cdots < X_{10}$ & $X_3 \in (y,y+\Delta y))$

$$\approx P(X_3 \in (y,y+\Delta y))\, P(X_1 < X_2 < y)\, P(y+\Delta y < X_4 < \cdots < X_{10})$$

$$\approx \Delta y \, \frac{y^2}{2!}\cdot\frac{(1-y)^7}{7!}$$

(c) the density of Y is $\frac{10!\,y^2(1-y)^7}{2!\,7!}$, which is the $B(\alpha,\beta)$ density with $\alpha = 3$, $\beta = 8$.

2. the pdf of $X+Y+Z$ is the convolution of the pdf of $X+Y$, which is

$$f_{X+Y}(t) = \begin{cases} t & 0\le t\le 1 \\ 2-t & 1<t\le 2,\end{cases}$$

and the pdf of Z. this is

$$f_{X+Y+Z}(t) = \int_{\mathbb{R}} f_{X+Y}(u)\, f_Z(t-u)\,du = \begin{cases} t^2/2 & 0\le t\le 1 \\ (-2t^2+6t-3)/2 & 1\le t\le 2 \\ (t-3)^2/2 & 2\le t\le 3\end{cases}$$

3. Skipped

4. $Y_1 = -\log X_1$ has the exponential density e^{-x},

So S_n has density

$$\frac{x^{n-1} e^{-x}}{(n-1)!}$$

So Z_n has density

$$\frac{1}{(n-1)!} \left(\log(1/x)\right)^{n-1}.$$

5. We convolve the density of X_1 and the density of $-X_2$:

$$f_Z(x) = \int_0^\infty e^{\lambda(x-2y)}\, dy = \frac{1}{2\lambda} e^{\lambda x} \quad \text{for } x<0,$$

and $f_Z(x) = \int_x^\infty e^{\lambda(x-2y)}\, dy = \frac{1}{2\lambda} e^{-\lambda x}$, $x \geq 0$.

So $f_Z(x) = \frac{1}{2\lambda} e^{-\lambda|x|}$.

6. It does not. It ensures that the average will not deviate by more than any fixed $\varepsilon > 0$ ~~as~~ with high probability for large enough n, but ~~it does~~ that says the actual count will be within $\frac{1}{2}n - \varepsilon n$ and $\frac{1}{2}n + \varepsilon n$, and $\varepsilon n \geq 100$ for large n.

7. $E(X) = 10$, $\operatorname{Var} X = \int_0^{20} (x-10)^2 \frac{dx}{20} = \frac{100}{3}$

$$P(|X-10| \geq k) = \frac{4}{5}, \frac{1}{2}, \frac{1}{10}, 0 \quad \text{for } k = 2, 5, 9, 20$$

Chebyshev gives $\frac{\sigma^2}{k^2} = \frac{100}{12}, \frac{100}{75}, \frac{100}{243}, \frac{100}{1200}$.

8. $$P(X \geq a) = \frac{1}{a} E(a \mathbb{1}_{X \geq a})$$
$$\leq \frac{1}{a} E(X),$$
since $a \mathbb{1}_{X \geq a} \leq X$ for all $\omega \in \Omega$.