Math 1610
Homework 5
Solutions

1. (a) $E(W) = \sum_{N} 2^N P(\text{first head on } N^{\text{th}} \text{ flip})$

$$= \sum_{N} 2^N 2^{-N}$$

$$= \sum_{N} 2^0 = +\infty$$

(b) $$E(W) = \sum_{N=1}^{29} 2^N 2^{-N} + \sum_{N=30}^{\infty} 2^{30} 2^{-N}$$

$$= 29 + 2 = 31$$

(c) $$E(U) = \sum_{N=1}^{\infty} \sqrt{2^N}\, 2^{-N} = \sum_{N=1}^{\infty} 2^{-N/2} = \frac{2^{-1/2}}{1 - 2^{-1/2}} = \frac{1}{\sqrt{2}-1}$$

2. Let $P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$. Then

$$E(X) = \sum_{k=0}^{\infty} \frac{k\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!}$$

$$= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \qquad \ell = k-1$$

$$= \lambda e^{-\lambda} \sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\ell!}$$

$$= \lambda.$$

3. $\mathrm{Var}(X+Y) = E((X+Y)^2)$

$$= EX^2 + EXY + EY^2$$

$$\mathrm{Var}(X-Y) = EX^2 - EXY + EY^2,$$

so $E(XY) = 0$. Also, $EX = E(X+Y) = E(X) + E(Y)$, so $E(X) = 0$ and similarly for $E(Y) = 0$.

Also,

$$\mathrm{Var}\,X = \mathrm{Var}(X+Y) \Rightarrow$$

$$EX^2 = EX^2 + EXY + EY^2 \Rightarrow$$

$$EY^2 = 0,$$

and same for $EX^2 = 0$. Now $EX^2 = 0 \Rightarrow X = 0$, since if $X \neq 0$ with prob. greater than zero, then we'd have $EX^2 > 0$.

4. $ES_n = \sum_{j=1}^{n} p_j = \frac{7}{10} n$ is our constraint on p_j. the variance of S_n is $\sum_{j=1}^{n} p_j(1-p_j)$. We can solve this constrained optimization problem using Lagrange multipliers: $f(\underline{p}) = \sum_{j=1}^{\hat{n}} p_j(1-p_j)$, $g(\underline{p}) = \sum_{j=1}^{\hat{n}} p_j$,

$$\nabla f = \lambda \nabla g \Rightarrow \langle 1-2p_1, \ldots, 1-2p_n \rangle = \langle \lambda p_1, \ldots, \lambda p_n \rangle \Rightarrow$$

$$p_1 = p_2 = \cdots = p_n.$$

5. Let X_1 be the index of the first head, X_2 the number of flips from the first to the second, etc. Then the desired random variable equals $X_1 + X_2 + \cdots + X_n$, and the X_i's are iid, $\mathrm{Geom}(p)$.

$$E(\text{index of } n^{\text{th}} \text{ head}) = E(X_1 + \cdots + X_n)$$

$$= EX_1 + \cdots + EX_n.$$

$$= \frac{1}{p} + \cdots + \frac{1}{p} = \frac{n}{p}.$$

and

$$\mathrm{Var}(X_1 + \cdots + X_n) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n)$$

$$= \frac{1-p}{p^2} + \cdots + \frac{1-p}{p^2}$$

$$= \frac{n(1-p)}{p^2}$$

6. We calculate $\phi'(a) = -2EX + 2a$, since $\phi(a) = EX^2 - 2aEX + a^2$, and $\phi'(a) = 0 \Rightarrow a = EX$.

7. $$E(X^Y) = \int_0^1 \int_0^1 x^y \, dx \, dy = \int_0^1 \frac{x^{y+1}}{y+1} \Big|_0^1 dy$$

$$= \int_0^1 \frac{1}{y+1} dy = \log(y+1) \Big|_0^1$$

$$= \log 2 - \log 1$$

$$= \boxed{\log 2}$$

8.(a) $\operatorname{cov}(X,Y) = E((X-\mu_X)(Y-\mu_Y))$

$$= EXY - \mu_X \overbrace{EY}^{=\mu_Y} - \mu_Y \overbrace{EX}^{=\mu_X} + \mu_X\mu_Y$$

$$= EXY - (EX)(EY).$$

(b) If X, Y are independent, then $E(XY) = E(X)E(Y)$, so the expression in (a) vanishes.

(c) $\operatorname{Var}(X+Y) = EX^2 + 2E(XY) + EY^2 - (EX)^2 - 2EXEY - (EY)^2$

$$= \operatorname{Var}X + \operatorname{Var}Y + 2\operatorname{cov}(X,Y)$$

9. (a) $\operatorname{Var}\left(\frac{X}{\sigma(X)} + \frac{Y}{\sigma(Y)}\right) = \operatorname{Var}\left(\frac{X}{\sigma(X)}\right) + \operatorname{Var}\left(\frac{Y}{\sigma(Y)}\right) + 2\operatorname{cov}\left(\frac{X}{\sigma(X)}, \frac{Y}{\sigma(Y)}\right)$ ~~$\operatorname{Var}\left(\frac{XY}{\sigma(X)\sigma(Y)}\right)$~~

$$= 1 + 1 + 2\frac{\operatorname{cov}(X,Y)}{\sigma(X)\sigma(Y)}$$

$$= 2 + 2\rho(X,Y).$$

(b) same as (a) but with a negative sign.

(c) $2(1+\rho(x,y)) \geq 0 \Rightarrow \rho(x,y) \geq -1$

$2(1-\rho(x,y)) \geq 0 \Rightarrow \rho(x,y) \leq 1.$

10. $f(x) = \int_0^1 f_{X,Y}(x,y)\,dy = \int_0^1 \overbrace{f_{X|Y}(x|y)}^{\frac{1}{y} \text{ on } [0,y]} \overbrace{f_Y(y)}^{1 \text{ on } [0,1]}\,dy = \int_x^1 \frac{dy}{y} = \boxed{-\log X}$