BROWN UNIVERSITY
Probability Math 1610
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Problem Set 9
Due: 3 December 2015 at 11:59 PM

Problem numbers refer to Grinstead & Snell.

1. Find the moment generating function of (a) Unif$([0, 1])$, (b) Exp(λ), and (c) $\mathcal{N}(0, 1)$. In each case, differentiate the mgf to find the mean and variance of the distribution.

Solution. (a) The mgf of the uniform distribution is

$$\int_0^1 e^{tx}\,dx = \frac{e^t - 1}{t}$$

The Taylor series expansion of the exponential function tells us that that this function equals $1 + t/2 + t^2/3! + \cdots$, and from this representation, we can see that the derivative at 0 equals $1/2$ and the second moment is at 0 is $1/3$. Thus the variance is $1/3 - (1/2)^2 = 1/12$.

(b) The mgf is

$$\int_0^\infty e^{tx}\lambda e^{-\lambda x}\,dx = \frac{\lambda}{\lambda - t}.$$

Differentiating and substituting $t = 0$ gives $1/\lambda$ for the mean and $2/\lambda^2$ for the second moment. Thus the variance is $2/\lambda^2 - (1/\lambda)^2 = 1/\lambda^2$.

(c) The mgf of the standard normal is

$$\int_{-\infty}^\infty e^{tx}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx = e^{t^2/2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-t^2/2}e^{tx}e^{-x^2/2}\,dx = e^{t^2/2},$$

since the integral is the total probability mass of the distribution $\mathcal{N}(t, 1)$ and is therefore equal to 1. Differentiating (or Taylor expanding), we find that the mean is zero and the second moment is 1. □

2. (#1 on p. 392) Let $Z_1, Z_2, \ldots, Z_N$ describe a branching process in which each parent has j offspring with probability p_j. Find the probability d that the process eventually dies out if

(a) $p_0 = 1/2$, $p_1 = 1/4$, and $p_2 = 1/4$.

(b) $p_0 = 1/3$, $p_1 = 1/3$, and $p_2 = 1/3$.

(c) $p_0 = 1/3$, $p_1 = 0$, and $p_2 = 2/3$.

(d) $p_j = 1/2^{j+1}$, for $j = 0, 1, 2, \ldots$.

(e) $p_j = (1/3)(2/3)^j$, for $j = 0, 1, 2, \ldots$.

3. The means of these five distributions are $3/4, 1, 4/3, 1$, and 2. Thus the extinction probabilities are 1, 1, less than 1, 1, and less than 1. To find the two that are less than one, we solve $h(z) = z$, where $h(z) = \sum p_k z^k$ is the probability generating function. In both cases, we find that the solution is $z = 1/2$.

4. (#4 on p. 392) Let $X_1, X_2, \ldots$ be i.i.d. random variables such that the probabiity generating function of X_1 is $f(z)$. Assume that N is an integer valued random variable independent of all the X_j's and having probability generating function $g(z)$. Show that the generating function for

$$S_N = X_1 + \cdots + X_N$$

is $g(f(z))$. Hint: use the fact that

$$h(z) = E(z^{S_N}) = \sum_{k=0}^{\infty} E(z^{S_N} \mid N = k) P(N = k).$$

Proof. We have

$$\begin{aligned} h(z) &= \sum_{k=0}^{\infty} E(z^{S_N} \mid N = k) P(N = k) \\ &= \sum_{k=0}^{\infty} E(z^{X_1} \cdots z^{X_k} | N = k) P(N = k) \\ &= \sum_{k=0}^{\infty} f(z)^N P(N = k). \end{aligned}$$

Thus $h(z) = g(f(z))$, as desired. □

5. (#7 on p. 393) Let N be the expected number of total offspring in a branching process (over all generations). Denote by m the mean number of offspring of a single parent. Show that $N = 1 + mN$. Conclude that N is finite if and only if $m < 1$, and solve for N in that case.

Proof. We condition on the number X of offspring in the first generation. Denote by D the number of descendants from the first generation forward, and note that

$$\begin{aligned} N &= 1 + E(D \mid X = 1) P(X = 1) + E(D \mid X = 2) P(X = 2) + \cdots \\ &= 1 + Np_1 + 2Np_2 + 3Np_3 + \cdots = 1 + mN. \end{aligned}$$

If $m \geq 1$, then there is no solution to this equation since the lines N and $1 + mN$ do not intersect. If $m < 1$, then there they intersect at $N = 1/(1 - m)$. □

6. (#10 on p. 403) Let $X_1, X_2, \ldots, X_n$ be an i.i.d. sequence of random variables so that X_1 has density $f(x) = \frac{1}{2} e^{-|x|}$ for $x \in \mathbb{R}$.

(a) Find the mean and variance of X_1.

Proof. The mean is zero by symmetry. The variance is obtained by integrating x^2 times the density, which is

$$\int_{-\infty}^{0} x^2 e^x / 2 \, dx + \int_{0}^{\infty} x^2 e^{-x} / 2 \, dx = 2,$$

where the integral is performed using integration by parts. □

(b) Find the moment generating function for X_1, S_n, A_n, and S_n^*.

Proof. The mgf is the integral of e^{xt} times the density, and it works out to $\dfrac{1}{1-t^2}$ (again, one splits the real line into its positive and negative parts). Thus the mgfs of S_n, A_n, and S_n^* are $(1-t^2)^{-n}$, $(1-(t/n)^2)^{-n}$, and $(1/(1-(t/2\sqrt{n})^2))^{-n}$, respectively. □

(c) What can you say about the convergence of the moment generating function of S_n^* as $n \to \infty$?

Proof. It converges to $e^{t^2/2}$, using the fact that $(1+x/n)^n$ converges to e^x as $n \to \infty$. The same result also follows from the central limit theorem. □

(d) What can you say about the convergence of the moment generating function of A_n as $n \to \infty$?

Proof. It converges to the constant function 1, for the same reason. □

7. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables distributed uniformly in $\{1,2,3,4,5,6\}$, which we'll think of as dice rolls. For each of the following sequences, determine whether it is a Markov chain, and if so, determine its transition probabilities.

(a) $(S_n)_{n=1}^{\infty}$, where S_n is the number of 6's rolled in the first n rolls.

Solution. This is a Markov chain with transition probabilities $P(n, n+1) = 1/6$ and $P(n,n) = 5/6$, for all n. □

(b) $(M_n)_{n=1}^{\infty}$, where M_n is the largest number thrown among the first n rolls.

Solution. This is a Markov chain with transition probabilities $P(n,m) = m/6$ whenever $m \geq n$ and $P(n,m) = 0$ when $m < n$. □

(c) $(R_n)_{n=1}^{\infty}$, where R_n is the number of times the result of the nth roll appeared among the first $n-1$ rolls.

Solution. This is not a Markov chain, because the conditional probability of the next value given the present value is different from the conditional probability of the next value given all previous values. We can prove this by example: if the first three values were 1, 2, 1, then the conditional probability that the next value is 3 would be 1/6. However, if the first three values were 1, 1, 1, then the conditional probability of getting a 3 next would be 0. □

8. Every second, a frog jumps from its current lilypad to a different one with probabilities as follows: from lilypad A, the frog jumps to B with probability 1/5 and C with probability 4/5. From lilypad B, the frog jumps to A with probability 3/5 and to C with probability 2/5. From lilypad C, the from jumps to B with probability 1/2 and to C with probability 1/2.

Write a transition matrix for a Markov chain modeling this situation, and raise it to the 5th power to find the probability that the frog is on lilypad C given that it started from lilypad A.

Solution.

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julia> P = [0 1//5 4//5; 3//5 0 2//5; 0 1//2 1//2]
3x3 Array{Rational{Int64},2}:
 0//1  1//5  4//5
 3//5  0//1  2//5
 0//1  1//2  1//2
julia> P^5
3x3 Array{Rational{Int64},2}:
  111//625    1903//6250    3237//6250
 2043//12500  1513//5000   13349//25000
    3//16     5849//20000  10401//20000
```

So the desired probability is 3237/6250. □

9. (#19 on p. 415) Consider the following process. We have two coins, one of which is fair, and the other of which has heads on both sides. We give these two coins to our friend, who chooses one of them at random (each with probability 1/2). During the rest of the process, she uses only the coin that she chose. She now proceeds to toss the coin many times, reporting the results. We consider this process to consist solely of what she reports to us.

(a) Given that she reports a head on the nth toss, what is the probability that a head is thrown on the $(n+1)$st toss?

Solution. The probability of two straight heads is $1/2$ (the probability that the double headed coin was selected) plus $1/8$ (the probability that the fair coin was selected and then two heads were flipped. The probability of the nth flip being heads is likewise $1/2 + 1/4$. Thus the conditional probability of heads on the $(n+1)$st flip given heads on the nth flip is

$$\frac{1/2 + 1/8}{3/4} = \frac{5}{6}.$$

□

(b) Consider this process as having two states, heads and tails. By computing the other three transition probabilities analogous to the one in part (a), write down a "transition matrix" for this process.

Solution. Repeating the above calculation for tails, we get

$$P = \begin{bmatrix} 5/6 & 1/6 \\ 1/2 & 1/2 \end{bmatrix}.$$

□

(c) Now assume that the process is in state "heads" on both the $(n-1)$st and the nth toss. Find the probability that a head comes up on the $(n+1)$st toss.

Solution. Modifying the calculation from the first part, we get

$$\frac{1/2 + 1/16}{1/2 + 1/8} = 9/10.$$

□

(d) Is this process a Markov chain?

Solution. The process is not a Markov chain, because the answers to (a) and (c) differ. □