Brown University

Probability Math 1610
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Problem Set 9
Due: 3 December 2015 at 11:59 PM

Problem numbers refer to Grinstead & Snell.

1. Find the moment generating function of (a) $\text{Unif}([0,1])$, (b) $\text{Exp}(\lambda)$, and (c) $\mathcal{N}(0,1)$. In each case, differentiate the mgf to find the mean and variance of the distribution.

2. (#1 on p. 392) Let $Z_1, Z_2, \ldots, Z_N$ describe a branching process in which each parent has j offspring with probability p_j. Find the probability d that the process eventually dies out if

(a) $p_0 = 1/2$, $p_1 = 1/4$, and $p_2 = 1/4$.

(b) $p_0 = 1/3$, $p_1 = 1/3$, and $p_2 = 1/3$.

(c) $p_0 = 1/3$, $p_1 = 0$, and $p_2 = 2/3$.

(d) $p_j = 1/2^{j+1}$, for $j = 0, 1, 2, \ldots$.

(e) $p_j = (1/3)(2/3)^j$, for $j = 0, 1, 2, \ldots$.

3. (#4 on p. 392) Let $X_1, X_2, \ldots$ be i.i.d. random variables such that the probabiity generating function of X_1 is $f(z)$. Assume that N is an integer valued random variable independent of all the X_j's and having probability generating function $g(z)$. Show that the generating function for

$$S_N = X_1 + \cdots + X_N$$

is $g(f(z))$. Hint: use the fact that

$$h(z) = E(z^{S_N}) = \sum_{k=0}^{\infty} E(z^{S_N} \mid N = k) P(N = k).$$

4. (#7 on p. 393) Let N be the expected number of total offspring in a branching process (over all generations). Denote by m the mean number of offpsring of a single parent. Show that $N = 1 + mN$. Conclude that N is finite if and only if $m < 1$, and solve for N in that case.

5. (#10 on p. 403) Let $X_1, X_2, \ldots, X_n$ be an i.i.d. sequence of random variables so that X_1 has density $f(x) = \frac{1}{2}e^{-|x|}$ for $x \in \mathbb{R}$.

(a) Find the mean and variance of X_1.

(b) Find the moment generating function for X_1, S_n, A_n, and S_n^*.

(c) What can you say about the convergence of the moment generating function of S_n^* as $n \to \infty$?

(d) What can you say about the convergence of the moment generating function of A_n as $n \to \infty$?

6. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables distributed uniformly in $\{1, 2, 3, 4, 5, 6\}$, which we'll think of as dice rolls. For each of the following sequences, determine whether it is a Markov chain, and if so, determine its transition probabilities.

(a) $(S_n)_{n=1}^{\infty}$, where S_n is the number of 6's rolled in the first n rolls.

(b) $(M_n)_{n=1}^{\infty}$, where M_n is the largest number thrown among the first n rolls.

(c) $(R_n)_{n=1}^{\infty}$, where R_n is the number of times the result of the nth roll appeared among the first $n-1$ rolls.

7. Every second, a frog jumps from its current lilypad to a different one with probabilities as follows: from lilypad A, the frog jumps to B with probability $1/5$ and C with probability $4/5$. From lilypad B, the frog jumps to A with probability $3/5$ and to C with probability $2/5$. From lilypad C, the from jumps to B with probability $1/2$ and to C with probability $1/2$.

Write a transition matrix for a Markov chain modeling this situation, and raise it to the 5th power to find the probability that the frog is on lilypad C given that it started from lilypad A. Note: you can enter matrices and raise them to powers in Julia as follows (here M is a 2×2 matrix):

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M = [0.2 0.8; 0.9 0.1]
M^10
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8. (#19 on p. 415) Consider the following process. We have two coins, one of which is fair, and the other of which has heads on both sides. We give these two coins to our friend, who chooses one of them at random (each with probability 1/2). During the rest of the process, she uses only the coin that she chose. She now proceeds to toss the coin many times, reporting the results. We consider this process to consist solely of what she reports to us.

(a) Given that she reports a head on the nth toss, what is the probability that a head is thrown on the $(n+1)$st toss?

(b) Consider this process as having two states, heads and tails. By computing the other three transition probabilities analogous to the one in part (a), write down a "transition matrix" for this process.

(c) Now assume that the process is in state "heads" on both the $(n-1)$st and the nth toss. Find the probability that a head comes up on the $(n+1)$st toss.

(d) Is this process a Markov chain?