BROWN UNIVERSITY

Probability Math 1610
Lead instructor: Samuel S. Watson
Problem Set 7
Due: 12 November 2015 at 11:59 PM

Problem numbers refer to Grinstead & Snell.

1. (#3 on p. 338) A true-false examination has 48 questions. June has probability 3/4 of answering a question correctly. April just guesses on each question. A passing score is 30 or more correct answers. Compare the probability that June passes the exam with the probability that April passes it.

2. (#4 on p. 338) Let S be the number of heads in 1,000,000 tosses of a fair coin. Use (a) Chebyshev's inequality, and (b) the Central Limit Theorem, to estimate the probability that S lies between 499,500 and 500,500. Use the same two methods to estimate the probability that S lies between 499,000 and 501,000, and the probability that S lies between 498,500 and 501,500.

3. (#9 on p. 339) Let S_n be the number of successes in n Bernoulli trials with probability 0.8 for success on each trial. Let $A_n = S_n/n$ be the average number of successes. In each case give the value for the limit, and give a reason for your answer.

(a) $\lim_{n\to\infty} P(A_n = 0.8)$

(b) $\lim_{n\to\infty} P(0.7n < S_n < 0.9n)$

(c) $\lim_{n\to\infty} P(S_n < 0.8n + 0.8\sqrt{n})$

(d) $\lim_{n\to\infty} P(0.79 < A_n < 0.81)$

4. (#17 on p. 340) In an opinion poll it is assumed that an unknown proportion p of the people are in favor of a proposed new law and a proportion $1 - p$ are against it. A sample of n people is taken to obtain their opinion. The proportion $\overline{p}$ in favor in the sample is taken as an estimate of p. Using the Central Limit Theorem, determine how large a sample will ensure that the estimate will, with probability 0.95, be correct to within 0.01.

5. (#2 on p. 354) A random walker starts at 0 on the x-axis and at each time unit moves 1 step to the right or 1 step to the left with probability 1/2. Estimate the probability that, after 100 steps, the walker is more than 10 steps from the starting position.

6. (#3 on p. 354) A piece of rope is made up of 100 strands. Assume that the breaking strength of the rope is the sum of the breaking strengths of the individual strands. Assume further that this sum may be considered to be the sum of an independent trials process with 100 experiments each having expected value of 10 pounds and standard deviation 1. Find the approximate probability that the rope will support a weight (a) of 1000 pounds, and (b) of 970 pounds.

7. Let[1] n be a fixed, large, even positive integer, and suppose that we flip a p-coin (a coin with probability $0 < p < 1$ of turning up heads and $1 - p$ of turning up tails) a total of n times. Denote by m the probability mass function of the resulting number of heads; note that m is a function from $\{0, 1, \ldots, n\}$ to $[0, 1]$. If $p = 1/2$, then m restricted to $\{0, 1, \ldots, n/2\}$ is an increasing function, and m restricted to $\{n/2, n/2 + 1, \ldots, n\}$ is a decreasing function. Do there exist values of $p \in (0, 1)$ such that m is an increasing function? How does this not contradict the central limit theorem?

[1] This problem is not in the book, and neither is the next one.

8. (A strong law of large numbers) Suppose that $X_1, X_2, \ldots$ are independent random variables with the property that $E(X_k) = 0$ for all k and $E(X_k^4) \leq K$ for all k, where $K > 0$ is a constant. Then

$$P\left(\lim_{n\to\infty} S_n/n = 0\right) = 1,$$

where $S_n = X_1 + X_2 + \cdots + X_n$.

(a) Write an expression for $E(S_n^4)$ by expanding out $(X_1 + \cdots + X_n)^4$. This may expression may look terrible at first, but most of the terms will vanish, because of the assumption that the random variables have zero mean.

(b) The fact that the variance of a random variable is nonnegative means that $E(Y^2) \geq (E(Y))^2$ for all random variables Y. Apply this observation to upper bound all the terms in the expression from part (a) by K. Conclude that

$$E(S_n^4) \leq 3Kn^2.$$

(c) Use (b) to conclude that $E\left(\sum_{n=1}^{\infty} (S_n/n)^4\right) < \infty$.

(d) Based on (c), what is the probability that $\sum_{n=1}^{\infty} (S_n/n)^4 = +\infty$? Explain why this implies $P\left(\lim_{n\to\infty} S_n/n = 0\right) = 1$.