BROWN UNIVERSITY

Probability Math 1610
Lead instructor: Samuel S. Watson
Problem Set 6
Due: 5 November 2015 at 11:59 PM

Problem numbers refer to Grinstead & Snell.

1. Let[1] $X_1, X_2, \ldots, X_{10}$ be independent random variables each of which has law Unif$([0,1])$. Define the random variable Y by letting $Y(\omega)$ be the third smallest number in the list $X_1(\omega), X_2(\omega), \ldots, X_{10}(\omega)$.

(a) Let Δt be a very small number. Show that $P(Y \in (y, y+\Delta y)) = 10! \cdot P(\{X_1 < X_2 < \cdots < X_{10}\} \cap \{X_3 \in (y, y+\Delta y)\})$, by considering all 10! possible orderings of the random variables $X_1, \ldots, X_n$.

(b) Calculate $P(\{X_1 < X_2 < \cdots < X_{10}\} \cap \{X_3 \in (y, y+\Delta y)\})$ by considering the probability that $\{X_3 \in (y, y+\Delta y)\}$, the probability that X_1 and X_2 are both less than y, the probability that $X_1 < X_2$ given that both are less than y, the probability that $X_4, \ldots, X_{10}$ are all greater than $y + \Delta y$, and the probability that $X_4 < X_5 < \ldots < X_{10}$ given that all these random variables are greater than $y + \Delta y$.

(c) Divide by Δy and take $\Delta y \to 0$ to find the probability density of Y. Show that Y is Beta distributed, and identify the Beta distribution parameters of the law of Y.

2. (#4 on p. 302) Find the pdf of $X + Y + Z$ where (X, Y, Z) is selected uniformly from the cube $[0,1]^3$.

3. (#7 on p. 310) Find the pdf of $X^2 + Y^2$ and the pdf of $\sqrt{X^2 + Y^2}$ where $X \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$.

4. (#13 on p. 303) Particles are subject to collisions that cause them to split into two parts with each part a fraction of the parent. Suppose that this fraction is uniformly distributed between 0 and 1. Following a single particle through several splittings we obtain a fraction of the original particle $Z_n = X_1 \cdot X_2 \cdot \cdots \cdot X_n$ where each X_j is uniformly distributed between 0 and 1. Show that the density for the random variable Z_n is $f_n(z) = \dfrac{1}{(n-1)!}(-\log z)^{n-1}$. Hint: Show that $Y_k = -\log X_k$ is exponentially distributed. Use this to find the density function for $S_n = Y_1 + Y_2 + \cdots + Y_n$, and from this the cumulative distribution and density of $Z_n = e^{-S_n}$.

5. (#14 on p. 303) Assume that X_1 and X_2 are independent random variables, each having an exponential density with parameter λ. Show that $Z = X_1 - X_2$ has density $f_Z(z) = (1/2)\lambda e^{-\lambda |z|}$.

6. (#8 on p. 313) A fair coin is tossed a large number of times. Does the Law of Large Numbers assure us that, if n is large enough, with probability exceeding 99%, the number of heads that turn up will not deviate from $n/2$ by more than 100?

7. (#2 on p. 321) Let X be a continuous random variable with values unformly distributed over the interval $[0, 20]$.

(a) Find the mean and variance of X.

(b) Calculate $P(|X - 10| \geq k)$ for $k \in \{2, 5, 9, 20\}$ exactly. How do your answers compare with the bounds you get for these quantities from Chebyshev's Inequality?

8. (#17 on p. 324) Show that, if $X \geq 0$, then $P(X \geq a) \leq E(X)/a$.

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[1] This problem is not from the book.