BROWN UNIVERSITY
Probability Math 1610
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Problem Set 3
Due: 6 October 2015

Problem numbers refer to Grinstead & Snell.

1. (#5 on p. 150) A coin is tossed three times. Consider the events $A =$ heads on first toss, $B =$ tails on the second toss, $C =$ heads on the third toss, $D =$ all three outcomes are the same, and $E =$ exactly one head turns up.

(a) Which of the following pairs are independent?

1. A, B
2. A, D
3. A, E
4. D, E

(b) Which of the following triples are independent?

1. A, B, C
2. A, B, D
3. C, D, E

2. (#7 on p. 151) A coin is tossed twice. Consider the events $A =$ heads on the first toss, $B =$ heads on the second toss, and $C =$ the two tosses come out the same.

(a) Show that A, B, C are pairwise independent but not independent.

(b) Show that C is independent of A and B but not of $A \cap B$.

3. (#18 on p. 152) A doctor assumes that a patient has one of three diseases d_1, d_2, or d_3. Before any test, he assumes an equal probability for each disease. He carries out a test that will be positive with probability 0.8 if the patient has d_1, 0.6 if he has disease d_2, and 0.4 if he has disease d_3. Given that the outcome of the test was positive, what probabilities should the doctor now assign to the three possible diseases?

4. Factory A and Factory B both make regular and softglow light bulbs. Factory A achieves lower bad-bulb rates for both regular bulbs and softglow bulbs. Is it possible that Factory B nevertheless achieves a lower overall bad-bulb rate?

Note: The bad-bulb rate is defined to be the number of bad bulbs produced divided by the total number of bulbs produced.

5. (#34 on p. 155) Four women, A, B, C, and D, check their hats, and the hats are returned in a random manner. Let Ω be the set of all possible permutations of A, B, C, D. Let $X_j = 1$ if the jth woman gets her own hat back and 0 otherwise. What is the distribution of X_j? Are the X_i's mutually independent?

6. (#39 on p. 156) Suppose that X and Y are random variables whose joint distribution is given by the following table.

Y \ X	−1	0	1	2
−1	0	1/36	1/6	1/12
0	1/18	0	1/18	0
1	0	1/36	1/6	1/12
2	1/12	0	1/12	1/6

(a) Find $P(X \geq 1 \text{ and } Y \leq 0)$.

(b) What is the conditional probability that $Y \leq 0$ given that $X = 2$?

(c) Are X and Y independent?

(d) What is the distribution of $Z = XY$?

7. (from #50 on p. 159) (a) Show that if $A_1, \ldots, A_n$ are mutually independent, then $B_1, \ldots, B_n$ are independent if for all i, we have $B_i = A_i$ or $B_i = \Omega \setminus A_i$. In other words, we may replace any number of events in an independent list with their complements, and the resulting list is still independent.

(b) Use (a) to show that, if $A_1, A_2, \ldots, A_n$ are independent events defined on a sample space Ω and if $0 < P(A_j) < 1$ for all $1 \leq j \leq n$, then Ω must have at least 2^n elements.

8. (#54 on p. 160) A deck of playing cards can be described as a Cartesian product

$$\text{Deck} = \text{Suit} \times \text{Rank},$$

where Suit $= \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$ and Rank $= \{2, 3, ..., 10, \text{J}, \text{Q}, \text{K}, \text{A}\}$. In other words, every card may be thought of as an ordered pair like $(\diamondsuit, 2)$, and every such ordered pair corresponds to a card. By a *suit event* we mean any event A contained in Deck which is described in terms of Suit alone.

For instance, if A is "the suit is red," then $A = \{\diamondsuit, \heartsuit\} \times \text{Rank}$, so that A consists of all cards of the form $(\diamondsuit, r)$ or $(\heartsuit, r)$ where r is any rank. Similarly, a *rank event* is any event described in terms of rank alone.

(a) Show that, under the uniform probability measure on Deck, if A is any suit event and B any rank event, then A and B are independent.

(d) Add a joker to the full deck, and consider uniform probability measure on the augmented deck $\text{Deck}^* = \text{Deck} \cup \{\text{Joker}\}$. Show that there is no pair A, B of nontrivial (that is, having probability strictly between 0 and 1) independent events. Hint: use the fact that 53 is prime.

9. (#2 on p. 172) A radioactive material emits an α-particle at a random time T whose distribution is given by the probability density function

$$f(t) = 0.1e^{-0.1t}.$$

Find the probability that the particle is emitted in the first ten seconds, given that

(a) no particle is emitted in the first second.

(b) no particle is emitted in the first 5 seconds.

(c) a particle is emitted in the first 3 seconds.

(d) a particle is emitted in the first 20 seconds.

10. (from #7 on p. 173) Let (X, Y) be chosen uniformly at random from the unit square $[0,1] \times [0,1]$. Show that the events $X \in [a,b]$ and $Y \in [c,d]$ are independent, for all $0 \le a \le b \le 1$ and $0 \le c \le d \le 1$.

11. (see #9 on p.174) Suppose that X and Y are random variables whose joint distribution is given by the density $f(x,y) = \frac{3}{2}(x^2 + y^2)\mathbf{1}_{x \in [0,1] \text{ and } y \in [0,1]}$ (that is, $f(x,y)$ equals $\frac{3}{2}(x^2 + y^2)$ if x and y are both between 0 and 1, and it equals zero otherwise). Find the probability density function of X.

12. (#12 on p. 174) Previous experience with a drug suggests that the probability p that the drug is effective is a random quantity having a beta density with parameters $\alpha = 2$ and $\beta = 3$. The drug is used on ten subjects and found to be successful in four out of the ten patients. What density should we now assign to the probability p? What is the probability that the drug will be successful the next time it is used?