BROWN UNIVERSITY

Probability Math 1610
Lead instructor: Samuel S. Watson
Due: 29 September 2015

Problem numbers refer to Grinstead & Snell.

1. (#7 on page 89) Five people get on an elevator that stops at five floors. Assuming that each person has an equal probability of going to any one floor, find the probability that they all get off at different floors.

2. (#8 on page 89) A finite set Ω has n elements. Show that there are 2^n subsets of Ω.

3. (#19 on page 90) Suppose that on planet Zorg a year has n days, and that the lifeforms there are equally likely to have hatched on any day of the year. We would like to estimate d, which is the minimum number of lifeforms needed so that the probability of at least two sharing a birthday exceeds $1/2$.

(a) In Example 3.3, it was shown that in a set of d lifeforms, the probability that no two life forms share a birthday is $(n)_d/n^d$. Thus, we would like to set this equal to $1/2$ and solve for d.

(b) Using Stirling's Formula, show that

$$\frac{(n)_d}{n^d} \sim \left(1 + \frac{d}{n-d}\right)^{n-d+1/2} e^{-d}.$$

(c) Now take the logarithm of the right-hand expression, and use the fact that for small values of x, we have

$$\log(1+x) \sim x - \frac{x^2}{2}.$$

(d) Set the expression in part (c) equal to $-\log 2$ to show that

$$d \sim \sqrt{(2\log 2)n}.$$

4. (#23 on page 92) Barbara Smith is interviewing candidates to be her secretary. As she interviews the candidates, she can determine the relative rank of the candidates but not the true rank. Thus, if there are six candidates and their true rank is 6, 1, 4, 2, 3, 5, (where 1 is best) then after she had interviewed the first three candidates she would rank them 3, 1, 2. As she interviews each candidate, she must either accept or reject the candidate. If she does not accept the candidate after the interview, the candidate is lost to her. She wants to decide on a strategy for deciding when to stop and accept a candidate that will maximize the probability of getting the best candidate. Assume that there are n candidates and they arrive in a random rank order.

(a) What is the probability that Barbara gets the best candidate if she interviews all of the candidates? What is it if she chooses the first candidate?

(b) Assume that Barbara decides to interview the first half of the candidates and then continue interviewing until getting a candidate better than any candidate seen so far. Show that she has a better than 25 percent chance of ending up with the best candidate.

5. (#6 on page 114) Charles claims that he can distinguish between beer and ale 75 percent of the time. Ruth bets that he cannot and, in fact, just guesses. To settle this, a bet is made: Charles is to be given ten small glasses, each having been filled with beer or ale, chosen by tossing a fair coin. He wins the bet if he gets seven or more correct.

Find the probability that Charles wins if he has the ability that he claims. Find the probability that Ruth wins if Charles is guessing.

6. (#14 on page 115) Use Stirling's formula to show that the probability that exactly n heads turn up in $2n$ tosses of a fair coin is asymptotic to $1/\sqrt{\pi n}$.

7. (#18 on page 115) Baumgartner, Prosser, and Crowell are grading a calculus exam. There is a true-false question with ten parts. Baumgartner notices that one student has only two out of the ten correct and remarks, "The student was not even bright enough to have flipped a coin to determine his answers." "Not so clear," says Prosser. "With 340 students I bet that if they all flipped coins to determine their answers there would be at least one exam with two or fewer answers correct." Crowell says, "I'm with Prosser. In fact, I bet that we should expect at least one exam in which no answer is correct if everyone is just guessing." Who is right?

8. (#22 on page 116) How many ways can six indistinguishable letters be put in three mail boxes? Hint: One representation of this is given by a sequence |LL|L|LLL| where the bars represent the partitions for the boxes and the L's the letters. Any possible way can be so described. Note that we need two bars at the ends and the remaining two bars and the six L's can be put in any order.

9. (#26 on page 116) You are playing heads or tails with Prosser but you suspect that his coin is unfair. Von Neumann suggested that you proceed as follows: Toss Prosser's coin twice. If the outcome is HT call the result win. If it is TH call the result lose. If it is TT or HH ignore the outcome and toss Prosser's coin twice again, continuing until you get either TH or HT. Repeat this procedure for each play. Assume that Prosser's coin turns up heads with probability p.

(a) Find the probability of HT, TH, HH, TT with two tosses of Prosser's coin.

(b) Using part (a), show that the probability of a win on any one play is $1/2$, no matter what p is.

10. (#30 on page 117) Recall that in the World Series the first team to win four games wins the series. The series can go at most seven games. Assume that the Red Sox and the Mets are playing the series. Assume that the Mets win each game with probability p. Note that, to simplify the analysis, we may pretend that all 7 games are played regardless of whether one team wins 4 games before the seventh game, and the winner is the team who wins 4 games or more.

(a) Using the program PowerCurve of Example 3.11 (available in the Binomial Probabilities Julia notebook on the course page), find the probability that the Mets win the series for the cases $p = 0.5$, $p = 0.6$, $p = 0.7$.

(b) Assume that the Mets have probability 0.6 of winning each game. Use the program PowerCurve to find a value of n so that, if the series goes to the first team to win more than half the games, the Mets will have a 95 percent chance of winning the series. Choose n as small as possible.

11. (#31 on page 117) Each of the four engines on an airplane functions correctly on a given flight with probability 0.99, and the engines function independently of each other. Assume that the plane can make a safe landing if at least two of its engines are functioning correctly. What is the probability that the engines will allow for a safe landing?

12. (#35 on page 118) Prove the following binomial identity

$$\binom{2n}{n} = \sum_{j=0}^{n} \binom{n}{j}^2.$$

Hint: Consider an urn with n red balls and n blue balls inside. Show that each side of the equation equals the number of ways to choose n balls from the urn.