MATH 520 MIDTERM II PRACTICE PROBLEMS
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Problem 1

Show that the set of $m \times n$ matrices, with the usual notion of matrix addition and scalar multiplication, is a vector space.

Solution

We check the ten conditions required of a vector space. The sum of two matrices is a matrix of the same dimensions (✓), and matrix addition inherits commutativity and associativity from real number addition (✓ and ✓). There is an additive identity, namely the matrix of all zeros (✓), and each matrix has an additive inverse: multiply each entry by -1 (✓).

A scalar times a matrix is another matrix of the same dimensions (✓), and scalar multiplication distributes across matrix addition again because of the corresponding property of real numbers, applied entry by entry (✓). Likewise, matrix multiplication distributes across *scalar* addition for the same reason (✓). The associative property holds for scalar-scalar-matrix multiplication, again by applying associativity of real number multiplication entry by entry (✓). Finally, one times a matrix is the same matrix (✓).

Problem 2

Show that for any n-dimensional vector space V and for any integer $0 \leq k \leq n$, there exists a subspace of V whose dimension is k.

Solution

Consider a basis $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ of V, which exists by the definition of finite-dimensional. Consider the span U of the list $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$. We know that U is a subspace of V because it is defined to be the span of a list of vectors in V. Furthermore, $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ is linearly independent since it is a sublist of a linearly independent list, and it spans U by definition. Therefore, $\mathcal{B}$ is a basis for U, and thus U is k-dimensional.

Problem 3

Suppose that V is a finite-dimensional vector space, U is a subspace of V, and W is a vector space. Suppose that $T : U \to W$ is a linear transformation. Show that there exists a linear transformation $\widetilde{T} : V \to W$ with the property that $T(\mathbf{v}) = \widetilde{T}(\mathbf{v})$ for all $\mathbf{v} \in U$.

Solution

The key idea here is that a linear transformation is determined by where it maps the vectors in a given basis of the domain space. These basis vectors can be mapped to *any* vectors in the codomain space. So we just need to identify a suitable basis of V which makes it easy to distinguish the vectors in U from the vectors in V which are not in U.

Consider a basis $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ of U, and extend it to a basis

$$\{\mathbf{b}_1, \ldots, \mathbf{b}_k, \mathbf{b}_{k+1}, \ldots, \mathbf{b}_n\}$$

of V. We're going to define $\widetilde{T}$ so that it agrees with T on U and maps all the rest of the basis vectors to $\mathbf{0}$ (this is chosen for simplicity; you can map them wherever you want). Each $\mathbf{v} \in V$ can be written uniquely as

$$\mathbf{v} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$$

by uniqueness of coordinates. We define $\widetilde{T}(\mathbf{v})$ to be $T(c_1\mathbf{b}_1 + \cdots + c_k\mathbf{b}_k)$. Then clearly $\widetilde{T}$ agrees with T on U, and $\widetilde{T}$ is linear since

$$\begin{aligned} T(\mathbf{v}+\mathbf{w}) &= T(c_1\mathbf{b}_1 + \cdots + c_k\mathbf{b}_k + d_1\mathbf{b}_1 + \cdots + d_k\mathbf{b}_k) \\ &= T(c_1\mathbf{b}_1 + \cdots + c_k\mathbf{b}_k) + T(d_1\mathbf{b}_1 + \cdots + d_k\mathbf{b}_k) \\ &= T(\mathbf{v}+\mathbf{w}), \end{aligned}$$

if the c's are the coordinates of $\mathbf{v}$ and the d's are the coordinates of $\mathbf{w}$. And similarly, we have $T(c\mathbf{v}) = cT(\mathbf{v})$.

Problem 4

Does there exist a basis of $\mathbb{P}_4$ such that none of the polynomials in the basis has degree 3?

Solution

Yes! For example,

$$\{t^4 + t^3, t^4 - t^3, t^2, t, 1\}.$$

This list spans $\mathbb{P}_4$ because any polynomial of degree 4 or less can be represented as a linear combination of these polynomials by using the coefficients of t^2, t, and 1 for the weights of the last three polynomials, and solving $a + b = c_4$ and $a - b = c_3$ for a and b to find the weights a and b for the first two polynomials. Since it spans and has length $\dim \mathbb{P}_4$, it is a basis.

Alternatively, we could show that the list above is a basis by writing down a homogeneous linear system that would have to be satisfied by any list of weights which annihilates this list of polynomials. We could row reduce to show that this linear system has only the trivial solution. Thus the list is linearly independent, and since it also has the right length, it must be a basis.

Problem 5

(Note: this problem involves calculus concepts, while the exam will not require such concepts more advanced than derivatives of polynomials).

Consider the map $T : C([0,1]) \to C([0,1])$ defined by

$$T(f) = g, \text{ where } g(x) = \int_0^x f(t)\,dt \text{ for all } x \in [0,1].$$

Show that T is linear, and describe the kernel and range of T.

Solution

The map T is linear since the integral is a linear operator (a basic fact from calculus). The kernel of T is the set of continuous functions whose antiderivative is the zero function, and the zero function is the only function with this property. So $\operatorname{Ker} T = \{\mathbf{0}\}$.

The range of T is the set of functions that can be obtained as the integral of a continuous function. Clearly $T(f)$ evaluates to 0 at 0, since the integral from 0 to 0 of any function is zero. Furthermore, $T(f)$ has a continuous derivative, since the derivative of $T(f)$ is f. So we conjecture that the range of T is the set of functions on $[0,1]$ which have continuous first derivative and which evaluate to 0 at 0.

This conjecture is correct: given such a function g, we can define $f = g'$, and then indeed $T(f) = g$ since $T(f) - g$ evaluates to 0 at 0 and has zero derivative (and is therefore the zero function).

Problem 6

Show that if T is a linear transformation from a vector space V to a vector space W, then $\dim T(V) \leq \dim V$.

Solution

The key idea is to show that the image of a basis of V under T spans $T(V)$.

Let $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis of V. We claim that $\{T(\mathbf{b}_1), \ldots, T(\mathbf{b}_n)\}$ is a spanning set for $T(V)$. To see this, suppose that $\mathbf{w} \in T(V)$, and consider $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{w}$. Write $\mathbf{v}$ as a linear combination of basis vectors:

$$\mathbf{v} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$$

Then $\mathbf{w}$ is indeed in the span of $\{T(\mathbf{b}_1), \ldots, T(\mathbf{b}_n)\}$ since linearity implies

$$T(\mathbf{v}) = c_1T(\mathbf{b}_1) + \cdots + c_nT(\mathbf{b}_n).$$

Since $T(V)$ has a spanning list of length n, its dimension is at most n.

Problem 7

Consider the set V of eventually-zero sequences of real numbers. A sequence is in V if and only if it has finitely many nonzero entries. For example,

$$(1, 2, 3, 0, 0, 0, 0, \ldots)$$

is in V, while

$$(1, 2, 0, 1, 2, 0, 1, 2, 0, \ldots)$$

is not. Show that V is a vector space, and show that it is infinite-dimensional.

Solution

To check that it is a vector space, we need to ensure that $\mathbf{v} + \mathbf{w} \in V$ whenever $\mathbf{v}, \mathbf{w} \in V$. Note that if $\mathbf{v}$ and $\mathbf{w}$ are both all zeros beyond the Nth entry, then so is $\mathbf{v} + \mathbf{w}$. Similarly, if $\mathbf{v}$ is eventually zero, then $c\mathbf{v}$ is eventually zero as well. The remaining vector space conditions can be checked the same way they're verified in the case of $\mathbb{R}^n$.

To see that V is infinite-dimensional, note that the vectors $\mathbf{e}_k$ with a 1 in position k and 0 elsewhere are in V and are linearly independent. So V has arbitrary long linearly independent lists, and since spanning lists are at least as long as linearly independent lists ($\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, for n as large as you like), this means that no finite list can span V.

Problem 8

Fix some vector $\mathbf{w} \in \mathbb{R}^n$. For $a \in \mathbb{R}$ and $\mathbf{u} \in \mathbb{R}^n$, define

$$a \otimes \mathbf{u} = a(\mathbf{u} - \mathbf{w}) + \mathbf{w}.$$

$$\mathbf{u} \oplus \mathbf{v} = \mathbf{u} + \mathbf{v} - \mathbf{w}.$$

Show that if we equip $V = \mathbb{R}^n$ with the operations $\otimes$ and $\oplus$ (as our scalar multiplication and vector addition, respectively), then we get a vector space. What is the zero vector in this new space? (Note: we use the symbols $\otimes$ and $\oplus$ to distinguish these notions of multiplication and addition from the usual ones in $\mathbb{R}^n$).

Solution

We check that V is closed under these new operations $\oplus$ and $\otimes$. This is true since evaluating the expressions $a(\mathbf{u} - \mathbf{w}) + \mathbf{w}$ and $\mathbf{u} + \mathbf{v} - \mathbf{w}$ always yields an element of $\mathbb{R}^n$.

To check additive commutativity:

$$\mathbf{v} \oplus \mathbf{u} = \mathbf{v} + \mathbf{u} - \mathbf{w} = \mathbf{u} \oplus \mathbf{v}.$$

Associativity works similarly.

The additive identity is actually the vector $\mathbf{w}$, since

$$\mathbf{u} \oplus \mathbf{w} = \mathbf{u} + \mathbf{w} - \mathbf{w} = \mathbf{u}.$$

The additive inverse of a vector $\mathbf{u}$ is obtained by solving the equation $\mathbf{u} \oplus \mathbf{v} = \mathbf{w}$ for $\mathbf{v}$ (we have $\mathbf{w}$ on the right-hand side of this equation since $\mathbf{w}$ is the additive identity in this new vector space). Thus we're solving

$$\mathbf{u} + \mathbf{v} - \mathbf{w} = \mathbf{w},$$

which tells us that the additive inverse of $\mathbf{u}$ is $-\mathbf{u} + 2\mathbf{w}$.

To check the first distributive property:

$$a \otimes (\mathbf{u} \oplus \mathbf{v}) = a \otimes (\mathbf{u} + \mathbf{v} - \mathbf{w}) = a(\mathbf{u} + \mathbf{v} - \mathbf{w} - \mathbf{w}) + \mathbf{w},$$

while

$$(a \otimes \mathbf{u}) \oplus (a \otimes \mathbf{v}) = a(\mathbf{u} - \mathbf{w}) + \mathbf{w} + a(\mathbf{v} - \mathbf{w}) + \mathbf{w} - \mathbf{w}.$$

The remaining properties may be checked similarly and are omitted.

Problem 9

Show that a linear transformation from a vector space V to W can be surjective only if $\dim W \leq \dim V$ and can be injective only if $\dim V \leq \dim W$.

Solution

If $\dim W > \dim V$, then $\dim T(V) \leq \dim V < \dim W$ by Problem 6, which means that $T(V)$ cannot be equal to W (since if they were equal they'd have the same dimension).

Conversely, if $\dim V > \dim W$, then if $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of V, the list $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ contains too many vectors to be linearly independent. Therefore, there is some nontrivial linear dependence relation

$$c_1T(\mathbf{v}_1) + \cdots + c_nT(\mathbf{v}_n) = \mathbf{0}$$

which implies by linearity that T maps $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ to the zero vector. Thus T is not injective.

Problem 10

Show that two vector spaces of the same finite dimension are isomorphic.

Solution

If V_1 and V_2 are both n-dimensional, then both are isomorphic to $\mathbb{R}^n$. This means that there exist isomorphisms $T_1 : V_1 \to \mathbb{R}^n$ and $T_2 : V_2 \to \mathbb{R}^n$. We can therefore define the composition $T_2^{-1} \circ T_1$ from V_1 to V_2. Since the composition of bijections is bijective, and since the composition of linear transformations is linear (exercise!), it follows that $T_2^{-1} \circ T_1$ is a bijective linear transformation from V_1 to V_2. Thus V_1 and V_2 are isomorphic.