Solution to Hw 9

#1. We know that $\det(A-\lambda I)=0$ if λ is an eigenvalue of A.

If $\lambda=0$, then we would have $\det(A-0\cdot I)=0$, hence $\det A=0$.

This is impossible, since A is invertible.

Therefore, $\lambda\neq 0$.

To see why $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}, let $\vec{v}$ be an eigenvector of A corresponding to λ.

By definition of eigenvalue and eigenvector, $A\vec{v}=\lambda\vec{v}$.

Multiply A^{-1} on both sides, we have $\underbrace{A^{-1}(A\vec{v})}_{(A^{-1}A)\vec{v}=\vec{v}} = \underbrace{A^{-1}\lambda\vec{v}}_{\lambda A^{-1}\vec{v}}$

Therefore, $\vec{v}=\lambda A^{-1}\vec{v}$,

So $A^{-1}\vec{v}=\frac{1}{\lambda}\vec{v}$. Hence $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}, and $\vec{v}$ is its corresponding eigenvector.

#2. Let A be an $n\times n$ matrix, each of whose rows has entry sum $= s$.

Then $$A\underbrace{\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}}_{n \text{ many } 1\text{'s}} = \begin{bmatrix}\text{sum of entries in } 1^{st} \text{ row of } A\\ \text{sum} \cdots\cdots 2^{nd} \cdots\\ \vdots \\ \text{sum} \cdots\cdots n^{th} \cdots\end{bmatrix} = \begin{bmatrix}s\\s\\\vdots\\s\end{bmatrix} = s\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}.$$

Since $A\vec{v}=s\vec{v}$ where $\vec{v}=\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}$, s is an eigenvalue of A, and $\vec{v}$ is its corresponding eigenvector.

\#3. This matrix is triangular. So $\det(A-\lambda I_8) = \underbrace{(1-\lambda)^2(4-\lambda)^4(2-\lambda)^2}_{\text{product of entries on the main diagonal}}$

The characteristic polynomial has 3 roots: $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = 2$.

When $\lambda_1 = 1$, the eigenspace is $\mathrm{Nul}(A - I_8)$.

$$A - I_8 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{3} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{3} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

Here the boxes represent pivot positions.

Then dimension of eigenspace $= \dim \mathrm{Nul}(A - I_8) = 8 - \underbrace{\mathrm{rank}(A - I_8)}_{\#\text{ of pivot columns}} = 2$

(alternatively, $\dim \mathrm{Nul}(A - I_8) = \#$ of free variables
$= \#$ of nonpivot columns.)

When $\lambda_2 = 4$,

$$A - 4 \cdot I_8 = \begin{bmatrix} \boxed{-3} & & & & & & & \\ & \boxed{-3} & & & & & 0 & \\ & & 0 & \boxed{1} & & & & \\ & & & 0 & \boxed{1} & & & \\ & & & & 0 & & & \\ & 0 & & & & 0 & & \\ & & & & & & \boxed{-2} & 1 \\ & & & & & & & \boxed{-2} \end{bmatrix}$$

So the eigenspace has dimension $8 - 6 = 2$.

When $\lambda_3 = 2$,

$$A - 2 \cdot I_8 = \begin{bmatrix} \boxed{-1} & & & & & & & \\ & \boxed{-1} & & & & 0 & & \\ & & \boxed{2} & 1 & & & & \\ & & & \boxed{2} & 1 & & & \\ & 0 & & & \boxed{2} & & & \\ & & & & & \boxed{2} & & \\ & & & & & & 0 & \boxed{1} \\ & & & & & & & 0 \end{bmatrix}$$

So the eigenspace has dimension $8 - 7 = 1$.

#4. Since $|A^n \vec{v}|$ converges to 0 as $n \to \infty$ for any nonzero $\vec{v}$, in particular $|A^n \vec{v}|$ must converge to 0 if $\vec{v}$ is an eigenvector of A.

Let $\vec{v_1}$ be an eigenvector corresponding to λ_1.

Then $A^n \vec{v_1} = A^{n-1}(A\vec{v_1}) = A^{n-1}\lambda_1\vec{v_1} = \lambda_1 A^{n-1}\vec{v_1}$

$= \lambda_1 A^{n-2} A\vec{v_1} = \lambda_1^2 A^{n-2}\vec{v_1}$

$= \cdots = \lambda_1^n \vec{v_1}$.

So $|A^n \vec{v_1}| = |\lambda_1|^n |\vec{v_1}|$

Since $\vec{v_1} \neq \vec{0}$, $|\vec{v_1}| \neq 0$. In order that $|A^n \vec{v_1}|$ converges to 0, we must have $\lim_{n\to\infty} |\lambda_1|^n = 0$.

This means $|\lambda_1| < 1$, so $-1 < \lambda_1 < 1$ (or $\lambda_1 \in (-1,1)$).

~~To show that~~

Similarly, replace $\vec{v_1}$ by $\vec{v_2}$ and λ_1 by λ_2, we get $-1 < \lambda_2 < 1$.

- - - - - - $\vec{v_3}$ - - - - - λ_3, - - - - $-1 < \lambda_3 < 1$.

To show that all values between -1 and 1 works,

write $A = PDP^{-1}$ (A is diagonizable since we have 3 distinct eigenvalues)

Then $|A^n \vec{v}| = |PD^nP^{-1}\vec{v}| = \left| P \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix} P^{-1}\vec{v} \right|$ for any $\vec{v} \neq \vec{0}$.

Now if $-1 < \lambda_1, \lambda_2, \lambda_3 < 1$,

then the matrix $D^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$ converges to zero matrix as $n \to \infty$,

hence $PD^nP^{-1}\vec{v}$ converges to zero vector.

Therefore, $\lim_{n\to\infty} |A^n \vec{v}| = 0$. if and only if $-1 < \lambda_1, \lambda_2, \lambda_3 < 1$.

#5. a). $\det(PDP^{-1})$

$= (\det P)(\det D)(\det P^{-1})$

$= (\det P)(\det P^{-1}) \det D$

$= \det(\underbrace{PP^{-1}}_{=I_n}) \det D = \det D = \det \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots & \lambda_n \end{bmatrix}$

$= \lambda_1 \lambda_2 \cdots \lambda_n.$

b). Write $\det(A - \lambda I_n) = c(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$.

We first show $c = (-1)^n$:

$$\det(A - \cancel{\lambda}\lambda I_n) = \begin{vmatrix} a_{11}-\lambda & * & \cdots & * \\ * & a_{22}-\lambda & * & \vdots \\ * & * & \ddots & * \\ * & * & \cdots * & a_{nn}-\lambda \end{vmatrix}$$

$= (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda) + (\text{polynomial of degree} \leq n-1)$

$= (-\lambda)^n + (\text{some other polynomial of degree} \leq n-1)$

$= (-1)^n \lambda^n + \cdots\cdots$

This means the leading coefficient of $c(\lambda-\lambda_1)\cdots(\lambda-\lambda_n) = \det(A - \lambda I_n)$ must be $(-1)^n$. Therefore $c = (-1)^n$.

Now $\det A = \det(A - 0 I_n) = (-1)^n (0-\lambda_1)(0-\lambda_2) \cdots (0-\lambda_n)$

$= (-1)^n (-1)^n \lambda_1 \cdots \lambda_n$

$= \lambda_1 \cdots \lambda_n.$