Math 520 - Homework 8 solutions.

1 (a) 3 bases for $\mathbb{P}_2$

- $\{1, t, t^2\}$
- $\{2, t-5, t^2\}$
- $\{1, t, t^2+t+1\}$

1 (b) 3 polynomials that don't form a basis

$$\{5+t, t^2, 5+t+t^2\}$$

2 $T: V \longrightarrow W$ Linear Transformation

Z subspace of W

show $T^{-1}(Z) = \{v \in V \mid T(v) \in Z\}$ is a subspace

(Note: this notation doesn't imply T is invertible!)

Sol'n

- Check: $0 \in T^{-1}(Z)$

  $T(0) = 0$, since Z subspace, $0 \in Z$

  Thus $0 \in T^{-1}(Z)$ ✓

- $x, y \in T^{-1}(Z) \Rightarrow x+y \in T^{-1}(Z)$

  $x, y \in T^{-1}(Z)$ means $T(x) \in Z$
  $T(y) \in Z$

  then $T(x+y) = T(x) + T(y)$

  but Z is a subspace, so if $T(x) \in Z$ and $T(y) \in Z$ then so is their sum. ✓

- $x \in T^{-1}(Z) \Rightarrow cx \in T^{-1}(Z)$ c scalar

  $x \in T^{-1}(Z)$ means $T(x) \in Z$

  $T(cx) = cT(x)$. since Z is a subspace it is closed under scaling. So $T(x) \in Z$ implies $cT(x) \in Z$ and so $cx \in T^{-1}(Z)$ ✓ $\square$

3 (a) $\{v_1 \cdots v_p\}$ spans V then
$\dim(V) \le p$.

TRUE We can shrink $\{v_1 \cdots v_p\}$
to $\{v_1 \cdots v_k\}$ which forms a
basis, so
$\dim(V) = k \le p$.

(b) $\{v_1 \cdots v_p\}$ linearly indep. then $\dim(V) \ge p$.

TRUE Can expand $\{v_1 \cdots v_p\}$ to
a basis $\{v_1 \cdots v_n\}$. Then
$\dim(V) = n \ge p$.

(c) If $\dim(V) = p$, there exists a
spanning list of $(p+1)$ vectors.

TRUE Take any basis of p vectors.
Add any other vector to it.
This new list has $(p+1)$ vectors and
still spans.

4 U, V 5-dimensional subspaces of $\mathbb{R}^9$

Show $\mathcal{U} \cap V \neq \{0\}$.

proof Suppose $\mathcal{U} \cap V = \{0\}$.

Let $\mathcal{B} = \{v_1 \cdots v_5\}$ be a basis for V.

$\mathcal{C} = \{u_1, \dots u_5\}$ be a basis for $\mathcal{U}$.

Claim: $\{v_1, \dots v_5, u_1, \dots, u_5\}$ is linearly indep.

Let $c_1 v_1 + \dots + c_5 v_5 + d_1 v_1 + \dots + d_5 v_5 = 0$

we will show all the c_i and d_i are $= 0$.

Rewrite this as

$$\underbrace{c_1 v_1 + \dots + c_5 v_5}_{\text{in } V} = \underbrace{-d_1 v_1 - \dots - d_5 v_5}_{\text{in } \mathcal{U}}$$

both sides are in $\mathcal{U}$ and V, since they are equal. Since $\mathcal{U} \cap V = \{0\}$ both sides are equal to 0.

$$c_1 v_1 + \cdots + c_s v_s = 0$$
$$d_1 v_1 + \cdots + d_s v_s = 0$$

then $c_1, \ldots, c_s$ are 0 b/c $\{v_1, \ldots, v_s\}$ is a basis.

Similarly, $d_1, \ldots, d_s$ are 0. //

Thus we see

$\{v_1, \ldots, v_s, u_1, \ldots, u_s\}$ are 10 linearly indep. vectors in $\mathbb{R}^9$. Impossible!

So we see $U \cap V \neq \{0\}$. $\square$

5 $\mathcal{U}$ nonzero subspace of V.

$T: V \longrightarrow W$ injective and linear.

Show $\dim(\mathcal{U}) = \dim(T(\mathcal{U}))$.

proof: let $\{u_1, \dots, u_k\}$ be a basis for $\mathcal{U}$. We'll see $\{T(u_1), \dots, T(u_k)\}$ is a basis for $T(\mathcal{U})$, so

$$\dim(\mathcal{U}) = k = \dim(T(\mathcal{U})).$$

Step 1 $\{T(u_1) \dots, T(u_k)\}$ spans $T(\mathcal{U})$.

let $w \in T(\mathcal{U})$. By definition, $w = T(u)$ for some vector $u \in \mathcal{U}$. Since $\{u_1, \dots, u_k\}$ is a basis for $\mathcal{U}$,

$$u = c_1 u_1 + \dots + c_k u_k$$

so

$$w = T(u) = T(c_1 u_1 + \dots + c_k u_k)$$
$$= c_1 T(u_1) + \dots + c_k T(u_k)$$

so w is a linear combination of $T(u_1)$ through $T(u_k)$.

<u>Step 2</u> $\{T(u_1) \ldots T(u_k)\}$ is linearly indep.

Let $c_1 T(u_1) + \cdots + c_k T(u_k) = 0$

then $T(c_1 u_1 + \cdots + c_k u_k) = 0$

but $T(0) = 0$ as well.

If T is injective,

$$0 = c_1 u_1 + \cdots + c_k u_k$$

since they have the same T-image.

But since $\{u_1 \cdots u_k\}$ is a basis,

$$c_1, \ldots, c_k = 0. \quad \square$$