SOLUTIONS TO PROBLEM SET 7 [MATH 520]

Please read this in conjunction with the corresponding problem set.

Problem 1. Statements a and c are false; while statements b, d, e, and f are true. Statement g is false since $(1,0,0)$ and $(0,1,0)$ are linearly independent in $\mathbb{R}^3$, but do not form a basis for it. Statement h is true since the span of any linearly independent list always yields a subspace (the addition and scalar multiplication conditions are satisfied).

Problem 2a. Let $A = \{f \in C([0,1]) : f(1/3) = 0 = f(2/3)\}$. Suppose $g, h \in A$. Since

$$(g+h)(1/3) = g(1/3) + h(1/3) = 0 + 0 = 0 = 0 + 0 = g(2/3) + h(2/3) = (g+h)(2/3),$$

and for all real scalars c we have

$$cg(1/3) = c \cdot 0 = 0 = c \cdot 0 = cg(2/3),$$

we can conclude that A is indeed a subspace of $C([0,1])$.

Now let $B = \{f \in C([0,1]) : f(1/3) = 0 \text{ or } f(2/3) = 0\}$. Suppose $g, h \in B$ such that

$$g(1/3) = 0 \neq g(2/3) \text{ and } h(1/3) \neq 0 = h(2/3).$$

Clearly

$$(g+h)(1/3) = g(1/3) + h(1/3) \neq 0 \neq g(2/3) + h(2/3) = (g+h)(2/3).$$

Hence $g + h \notin B$ and thus B is not a subspace of $C([0,1])$.

Problem 2b. No. Denote the subset of $\mathbb{P}_5$ consisting of all polynomials of degree 5 by P_5. Take $p = t^5 + 4t^4$ and $q = -t^5$. We see that $p, q \in P_5$. However $p + q \notin P_5$. Therefore it is not a subspace.

Problem 3. Suppose $\{1, \sin(t), \sin(2t), \sin(3t)\}$ is a linearly dependent set in $C([0,1])$. Then there exist real scalars c_0, c_1, c_2, c_3 not all zero such that

$$c_0 + c_1 \sin(t) + c_2 \sin(2t) + c_3 \sin(3t) = 0$$

for all $t \in [0,1]$. Substituting $t = 0.1, 0.2, 0.3, 0.4$ into this equation gives a system of equations which we represent by the 4x4 matrix

$$\begin{bmatrix} 1 & \sin(0.1) & \sin(0.2) & \sin(0.3) \\ 1 & \sin(0.2) & \sin(0.4) & \sin(0.6) \\ 1 & \sin(0.3) & \sin(0.6) & \sin(0.9) \\ 1 & \sin(0.4) & \sin(0.8) & \sin(1.2) \end{bmatrix}$$

By using Julia, Mathematica, or any other software program one finds that the determinant is nonzero. Specifically you'll get a tiny value of order of magnitude -6:

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In[13]:= mat = {{1, Sin[0.1], Sin[0.2], Sin[0.3]}, {1, Sin[0.2], Sin[0.4], Sin[0.6]},
           {1, Sin[0.3], Sin[0.6], Sin[0.9]}, {1, Sin[0.4], Sin[0.8], Sin[1.2]}}

Out[13]= {{1, 0.0998334, 0.198669, 0.29552}, {1, 0.198669, 0.389418, 0.564642},
          {1, 0.29552, 0.564642, 0.783327}, {1, 0.389418, 0.717356, 0.932039}}

In[15]:= MatrixForm[mat]
Out[15]//MatrixForm=
          1 0.0998334 0.198669  0.29552
          1  0.198669 0.389418 0.564642
          1   0.29552 0.564642 0.783327
          1  0.389418 0.717356 0.932039

In[14]:= Det[mat]

Out[14]= -3.24759 × 10^-6
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This contradicts our supposition. Thus $\{1, \sin(t), \sin(2t), \sin(3t)\}$ is linearly independent in $C([0,1])$.

Problem 4. $T(\vec{u}) = T(\vec{v}) \Rightarrow \vec{0} = T(\vec{u}) - T(\vec{v}) = T(\vec{u} - \vec{v}) \Rightarrow \vec{u} - \vec{v} = \vec{0} \Rightarrow \vec{u} = \vec{v}$.

Problem 5a. Suppose $\vec{w} \in V$ is not in the span of the linearly independent list $\{\vec{v}_1, ..., \vec{v}_n\}$. Then $\{\vec{v}_1, ..., \vec{v}_n, \vec{w}\}$ is a linearly independent set in V. But this contradicts the assumption that V is an n-dimensional vector space. Therefore the original list is also a basis.

Problem 5b. Suppose $\{\vec{v}_1, ..., \vec{v}_n\}$ spans V and is linearly dependent. By the linear dependence lemma there is some $\vec{v}_j$ in this list that is in the span of $\{\vec{v}_1, ..., \vec{v}_{j-1}, \vec{v}_{j+1}, ..., \vec{v}_n\}$. We can remove such a vector to get a list of $n - 1$ vectors that also spans V. But this contradicts the assumption that V is an n-dimensional vector space. Therefore the original list is also a basis.