MATH 520 PROBLEM SET 7
SPRING 2017
BROWN UNIVERSITY
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This problem set is due at the end of the day on Thursday, 23 March 2017. Please write up your solutions legibly, ***scan them*** *(either using a scanning app as per the instructions on the course website, or using an actual scanner—no plain photos), and upload them using Gradescope. There are also MyMathLab problems due at the same time.*

1 Indicate whether the following statements are true or false. Recall that you can only take the span of a list of vectors, the column space or null space of a matrix, and the kernel or range of a linear transformation. Assume A is an $m \times n$ matrix, and V is a vector space. For the last two, explain your answer.

(a) It makes sense to talk about the span of a matrix.

(b) It makes sense to talk about the span of the columns of a matrix.

(c) It makes sense, strictly speaking, to talk about the span of a vector.

(d) It makes sense to talk about the span of a list containing only one vector (hint: you might want to revise your answer to (c)).

(e) The column space of a matrix is a subspace of $\mathbb{R}^m$

(f) The null space of a matrix A and the kernel of the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ are both subspaces of $\mathbb{R}^n$.

(g) A linearly independent list of vectors in V is always a basis of V.

(h) A linearly independent list of vectors in V is always a basis of some subspace of V.

2 (a) Show that

$$\{f \in C([0,1]) \,:\, f(1/3) = 0 \text{ and } f(2/3) = 0\}.$$

is a subspace of $C([0,1])$. Show that

$$\{f \in C([0,1]) \,:\, f(1/3) = 0 \text{ or } f(2/3) = 0\}.$$

is not a subspace of $C([0,1])$.

(b) Is the subset of $\mathbb{P}_5$ consisting of all polynomials of degree 5 a subspace of $\mathbb{P}_5$? Recall that $\mathbb{P}_5$ is the space of polynomials of degree at most 5, with usual polynomial addition and scalar multiplication.

3 [some computational assistance necessary] Show that $\{1, \sin t, \sin 2t, \sin 3t\}$ are linearly independent in $C([0,1])$ by assuming there exist c_0, c_1, c_2, c_3—not all zero—such that $c_0 + c_1 \sin t + c_2 \sin 2t + c_3 \sin 3t = 0$ for all $t \in [0,1]$. Then substitute $t = 0.1, 0.2, 0.3, 0.4$ into this equation to find a system of equations satisfied by c_0, c_1, c_2, c_3. Reach a contradiction.

4 Consider a linear transformation T from one vector space V to another vector space W. Suppose that only vector $\mathbf{u} \in V$ for which $T(\mathbf{u}) = \mathbf{0}$ is $\mathbf{u} = \mathbf{0}$. Show that $T(\mathbf{u}) = T(\mathbf{v})$ implies $\mathbf{u} = \mathbf{v}$. In other words, this problem is asking you to show that if the kernel of T is $\{\mathbf{0}\}$, then T is injective.

5 (a) Show that if V is an n-dimensional vector space, and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of V. Hint: suppose for the sake of contradiction that there is a vector $\mathbf{w} \in V$ which is not in the span of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, and consider the list $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \mathbf{w}\}$ to obtain a contradiction.

(b) Show that if V is an n-dimensional vector space, and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ spans V, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of V. Hint: assume for sake of contradiction that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly dependent, and apply the linear dependence lemma (see handwritten course notes) to remove a vector from V. Use this shorter list to obtain a contradiction.