Math 520 Spring 2017
Solution to Problem Set 6

1) In each parts, we know that the area is given by the absolute value of the determinant of the matrix respectively.

   (a) $det\begin{bmatrix}1 & 2\\0 & 2\end{bmatrix} = 1 \ \cdot 2 - 0 \cdot 2 = 2$
   (b) $\det\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = 0 \cdot 0 - 1 \cdot 1 = \ -1$
   (c) $\det\begin{bmatrix}0 & 2\\-2 & 0\end{bmatrix} = 0 \cdot 0 - (-2) \cdot 2 = 4$
   (d) $\det\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} = 1 \cdot 1 - 1 \cdot 1 = 0$
   (e) $\det\begin{bmatrix}2 & -4\\-1 & 3\end{bmatrix} = 2 \cdot 3 - (-1) \cdot (-4) = 2$

2) If you plot out the quadrilateral directly, it is not hard to see that it is a parallelogram, but the vertices are not at the origin. To find the area, we translate it to the origin by subtracting the vector (2,2) to each of the vertices to obtain a parallelogram with vertices (0,0), (3,4), (1,2),(4,6), whose area is obtain by the absolute value of the determinant of
$$\begin{bmatrix}3 & 1\\4 & 2\end{bmatrix}$$
$$\det\begin{bmatrix}3 & 1\\4 & 2\end{bmatrix} = 3 \cdot 2 - 4 \cdot 1 = 2.$$
   Therefore the area of the required quadrilateral is 2.

3) Thinking of $A^T A$ as a composition of linear transformations
$$\mathbb{R}^5 \xrightarrow{A} \mathbb{R}^3 \xrightarrow{A^T} \mathbb{R}^5,$$
   the map A cannot be injective for the trivial dimension reasoning, so does $A^T A$. Therefore, from one of the equivalences in the inverse matrix theorem, we conclude that $A^T A$ is not invertible and hence $\det(A^T A) = 0$.

4) (a) As instructed, after the row operations you perform, the matrix looks like

$$\begin{bmatrix} a-b & b-a & 0 & \cdots & 0 \\ 0 & a-b & b-a & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a-b & b-a \\ b & \cdots & b & b & a \end{bmatrix}.$$

(b) After you perform the column operations, you should obtain

$$\begin{bmatrix} a-b & 0 & 0 & \cdots & 0 \\ 0 & a-b & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a-b & 0 \\ b & \cdots & (n-2)b & (n-1)b & a+(n-1)b \end{bmatrix}.$$

These column operations doesn't change the determinant as they can be think of as corresponding row operations of its transpose, which shares the same determinant.

(c) The matrix we obtained in part (b) is lower triangular, so that its determinant is exactly the product of its diagonal entries, which equals

$$(a-b)^{n-1}(a+(n-1)b)\,.$$

5) One choice of the labelling of the grid is

L	D	P	H
C	K	G	O
J	B	N	F
A	I	E	M

and the signs between the squares are assigned so that the Kasteleyn matrix is given by

$$K = \begin{array}{c} \\ I \\ J \\ K \\ L \\ M \\ N \\ O \\ P \end{array} \begin{array}{c} \begin{array}{cccccccc} A & B & C & D & E & F & G & H \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

We computed that its determinant is 36.