MATH 520 PROBLEM SET 6
SPRING 2017
BROWN UNIVERSITY
SAMUEL S. WATSON

*This problem set is due at the end of the day on Wednesday, 15 March 2017. Please write up your solutions legibly, (**starting a new page for each problem**), scan them, and upload them using Gradescope (submission instructions on the course website). There are also MyMathLab problems due at the same time.*

$\boxed{1}$ The following question is from Problem Set 4. Redo it using determinants. (Note: it is going to be way easier this time.)

For each of the following matrices, sketch the image of the unit square (the one bounded by the axes and the lines $x = 1$ and $y = 1$) under the linear transformation represented by the matrix. Calculate the area of each of these resulting regions.

(a) $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$

$\boxed{2}$ Use determinants to find the area of the quadrilateral whose vertices are $(2, 2)$, $(5, 6)$, $(3, 4)$ and $(6, 8)$.

$\boxed{3}$ Show that if A is a 3×5 matrix, then $\det(A^T A) = 0$. Note: you can't distribute the det, because A and A^T are not square!

$\boxed{4}$ (from #16 on p. 189 in your book) Let J be the $n \times n$ matrix all of whose entries are 1, and consider $A = (a - b)I + bJ$, which is the matrix whose diagonal entries are all a and the rest of whose entries are b. Show that

$$\det A = (a - b)^{n-1}(a + (n - 1)b)$$

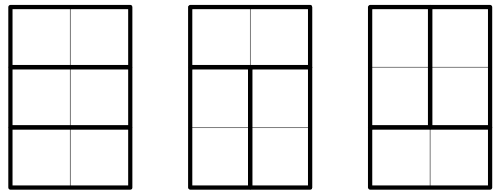
as follows.

(a) Subtract row 2 from row 1, row 3 from row 2, and so on. We know that these operations do not change the determinant.

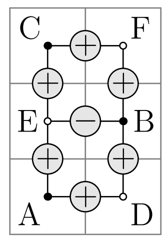
(b) With the result from (a), add column 1 to column 2, then add this new column 2 to column 3, and so on. Why do these steps not change the determinant?

(c) Find the determinant of the matrix you got in (b).

$\boxed{5}$ In this problem, we learn how to count domino tilings using determinants (!!). Consider a 3×2 grid. There are 3 ways to tile this grid using 2×1 dominos.



Here's another way, aside from trial and error, to figure that out (due to Kasteleyn, in the 1960s).



1. Label all the squares of the grid, and identify them as white or black in a chessboard pattern (shown with small white and black dots in the figure above.
2. Assign signs to each pair of adjacent squares in such a way that when you make a complete loop around four squares meeting at a corner, you encounter an odd number of negative signs. Such a choice of signs is indicated in the figure for the 3×2 grid (for example, in making the loop from E to B to F to C to E, we encounter one negative sign).
3. Form a matrix K with row labels given by the white square names and column labels given by the black square names.
4. Fill in the matrix entry in the (i, j)th position with the sign ($+1$ or -1) from the above figure if squares i and j are adjacent, or 0 if not.
5. Find the absolute value of $\det K$.

In the above example, we have

$$K = \begin{array}{c} \\ D \\ E \\ F \end{array} \!\! \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

and sure enough:

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julia> abs(det([1 1 0; 1 -1 1; 0 1 1])) == 3
true
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Use the above method to determine the number of domino tilings of the 4×4 grid below. (Hints: (i) there are many ways to choose the signs in step 2; just go at it and you'll probably find one pretty quickly, (ii) you'll have to be careful, as your matrix is going to have 64 entries in it, (iii) you'll probably want to use some computational assistance to evaluate the determinant of your Kasteleyn matrix.)