# MATH 520 HW5

#1. $C = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}$ (Starting your trial from the column $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$ may be more effective).

$CA = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$AC = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 3 & -1 \\ -2 & 4 & -1 \\ -4 & 6 & -1 \end{bmatrix} \neq I_3.$

#2. (a). percentage error $= \left|\frac{\Delta x}{x}\right| \times 100\%$ ($\begin{bmatrix} 3.94 \\ 0.49 \end{bmatrix}$ and $\begin{bmatrix} 2.9 \\ 2 \end{bmatrix}$ are solutions to these two linear systems)

percentage error for $x_1 = \left|\frac{2.9-3.94}{3.94}\right| \times 100\% \approx 26.4\%$

percentage error for $x_2 = \left|\frac{2-0.49}{0.49}\right| \times 100\% \approx 308.2\%$

(b). $\det A = \begin{vmatrix} 4.5 & 3.1 \\ 1.6 & 1.1 \end{vmatrix} = 4.5 \times 1.1 - 1.6 \times 3.1 = -0.01.$

(c). If $\det A = 0$, A is not invertible, so there can't be pivot position in every column. Consequently the solution can't be unique. This means the linear system is either inconsistent or has infinitely many solutions.

In this specific example, the system is inconsistent, because

$$\left[\begin{array}{cc|c} 4.5 & 3.1 & 19.249 \\ 1.6 & 1.10202 & 6.843 \end{array}\right] = \left[\begin{array}{cc|c} a & b & s \\ c & d & t \end{array}\right] \sim \left[\begin{array}{cc|c} ac & bc & cs \\ ac & ad & at \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} ac & bc & cs \\ 0 & \underset{\substack{\uparrow \\ \text{since } bc=ad}}{0} & cs-at \end{array}\right].$$

But $cs - at = 1.\underline{6} \times 19.24\underline{9} - 4.\underline{5} \times 6.84\underline{3} \neq 0.$
(last digit is 4) (last digit is 5)

(d). In this specific example, one can compute that the percentage error of $x_1$ is $\approx 0.03\%$, which is much smaller compare to the result in (a).

In general, there's a more geometric interpretation on this:

We know that solving $[\vec{v_1}\ \vec{v_2} \ \vdots\ \vec{b}]$ is to find weights $x_1, x_2$ such that $\vec{b} = x_1\vec{v_1} + x_2\vec{v_2}$. If $\det[\vec{v_1}\ \vec{v_2}]$ is close to $0$, $\vec{v_1}$ and $\vec{v_2}$ almost point in the same direction. Then any slight change in $\vec{b}$ would result in a relatively large percentage error in $x_1$ (try to decompose $\vec{b}$ and $\vec{b}'$ in the direction of $\vec{v_1}$ using parallelogram rule). See figure 1.

However, if we rotate $\vec{v_1}$ so that $\vec{v_1}$ makes a larger angle with $\vec{v_2}$ (this means $\det[\vec{v_1}\ \vec{v_2}]$ is farther from $0$), the change in $x_1$ is not so sensitive to the change in $\vec{b}$. See figure 2.

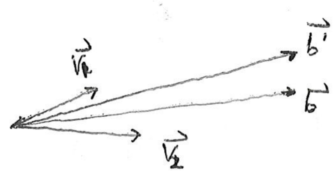

figure 1

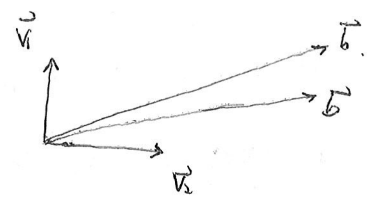

figure 2.

From this problem, we can see that 0 determinant is not the only situation we want to avoid. In fact, a determinant that is very close to 0 would also cause a problem, since the method of rounding off approximation would become less effective.

# 3. Since the columns of A are both linearly independent and span $\mathbb{R}^m$, there must be pivot position in every column and every row of A. So $m=n$.

#4. A $4\times5$ matrix has 5 columns. By rank theorem,

$$\text{rank} A + \dim \text{Nul} A = 5.$$

So $\dim \text{Nul} A = 3$ means $\text{rank} A = 5-3 = 2$.

#5. This is a consequence of basis theorem. (applied to the subspace Col A).

If A has rank 4, then any linearly independent set containing 4 elements must be a basis for Col A.