MATH 520 PROBLEM SET 4
SPRING 2017
BROWN UNIVERSITY
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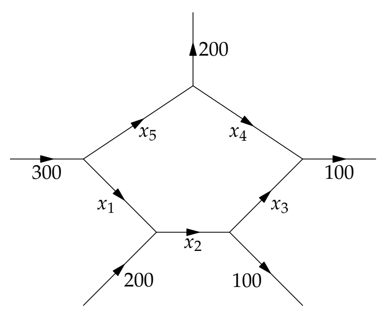
*This problem set is due at the end of the day on Thursday, 23 February 2017. Please write up your solutions legibly, (**starting a new page for each problem**), scan them, and upload them using Gradescope (submission instructions on the course website). There are also MyMathLab problems due at the same time.*

1 Suppose that a, b, and c are distinct real numbers. Show that the vectors $(1, 1, 1)$, (a, b, c), and (a^2, b^2, c^2) are linearly independent. (You may think of these as column vectors; we will often print vectors horizontally for typographical convenience.) Hint: set up an appropriate matrix with variables in it, and row reduce it.

2 Consider the traffic flow diagram below. Each street is one-way, and each label indicates the number of vehicles that traverse the given stretch of road during a fixed period of time (before and after which all the roads are clear).



(a) Show that one of the numerical labels must be incorrect, that is, there are no values of $x_1, \ldots, x_5$ which make the inflow and outflow at each intersection balance. Note: you may use computational assistance to row reduce, without needing to show intermediate steps. But you should explain your setup and reasoning.

(b) Show that if one of the numerical labels in the figure is changed suitably, then there are many possible solutions $(x_1, \ldots, x_5)$.

3 For each of the following matrices, sketch the image of the unit square (the one bounded by the axes and the lines $x = 1$ and $y = 1$) under the linear transformation represented by the matrix. Calculate the area of each of these resulting regions. Note: (f) isn't from the plane to the plane! You'll have to do a little 3D geometry thinking, but it isn't too bad.

(a) $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$

4 (based on #34 in Section 1.8) Suppose that T is a linear transformation and that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent list of vectors with the property that $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)\}$ is linearly dependent.

Show that the equation $T(\mathbf{x}) = \mathbf{0}$ has nontrivial solutions. Hint: write down an equation using the definition of linear dependence for $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)\}$. Then look to apply linearity of T.

5 Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^5$ and $S : \mathbb{R}^5 \to \mathbb{R}^3$ are linear transformations. Reason about the reduced row echelon form of the matrix of T and the matrix of S to answer the following questions (and include your reasoning).

(a) Is it possible that T is surjective?

(b) Is it possible that T is injective?

(c) Is it possible that S is surjective?

(d) Is it possible that S is injective?