(1) Suppose A has 4 rows and 3 columns, and suppose $b \in \mathbb{R}^4$. If $Ax = b$ has exactly one solution, what can you say about the reduced row echelon form of A? Explain.

Solution: If $Ax = b$ has exactly one solution, there cannot be any free variables in this system. Since free variables correspond to non-pivot columns in the reduced row echelon form of A, we deduce that every column is a pivot column. Thus the reduced row echelon form must be

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(2) Indicate whether each statement is true or false. If it is false, give a counterexample.
(a) The vector b is a linear combination of the columns of A if and only if $Ax = b$ has at least one solution.
(b) The equation $Ax = b$ is consistent **only** if the augmented matrix $[A \mid b]$ has a pivot position in each row.
(c) If matrices A and B are row equivalent $m \times n$ matrices and $b \in \mathbb{R}^m$, then the equations $Ax = b$ and $Bx = b$ have the same solution set.
(d) If A is an $m \times n$ matrix whose columns do not span $\mathbb{R}^m$, then the equation $Ax = b$ is inconsistent for some $b \in \mathbb{R}^m$.

Solution: (a) True, as Ax can be expressed as a linear combination of the columns of A via the coefficients in x.
(b) Not necessarily. Suppose A is the zero matrix and b is the zero vector. Then any x satisfies the equation $Ax = b$ but there are no pivot positions at all.
(c) This is not generally true, as when applying operations to the left-hand side of the augmented matrix $[A \mid b]$ the vector b on the right-hand side will change as well. So even if A is equivalent to B, the augmented system $[A \mid b]$ and $[B \mid b]$ need not be equivalent. For example, take A to be the 1×1 matrix $A = [3]$ and $B = [1]$ its echelon form. Supposing that b is the vector $b = [3]$. Then we are asking if the following two equations have the same solution set

$$3x = 3$$

$$1x = 3$$

which is clearly not the case.
(d) This is true, just take b to be a vector not in the span of the columns. Then $Ax = b$ is inconsistent by part (a).

(3) Suppose

$$A = \begin{pmatrix} -2 & 4 & -7 & 5 \\ -7 & 1 & 6 & -2 \\ -6 & 5 & 1 & 0 \\ 3 & -2 & 5 & 6 \\ 4 & -5 & -5 & -6 \end{pmatrix}$$

Then $x = (1, 1, 1, 1)$ is in the solution set of the equation $Ax = 0$. Use this information to find nonzero constants c_1, c_2, c_3, c_4 such that $c_1v_1 + \cdots + c_4v_4 = 0$, where v_1, v_2, v_3, v_4 are the columns of A.

Solution: This problem demonstrates how a solution to $Ax = 0$ provides a dependence relation between the columns of A. If we write

$$A = [v_1 \;\; v_2 \;\; v_3 \;\; v_4]$$

then the expression $Ax = 0$ can be rewritten as

$$1v_1 + 1v_2 + 1v_3 + (-1)v_4 = 0$$

The coefficients $c_1, \cdots, c_4$ are the just the entries of x!

(4) Consider the vectors

$$u = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \quad w = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

(a) Verify, by showing that they do not span $\mathbb{R}^3$, that these vectors are coplanar.
(b) Write down a system of equations whose solution set is equal to span$\{u, v, w\}$.
(c) The blue plane in the figure is parallel to the green one and passes through the point $(0, 0, 3)$. Use what you know about systems of homogeneous linear equations to alter the system you gave as an answer

in (b) so that the solution set of your new system is equal to the blue plane.

Solution: (a) We can build a matrix A from these vectors,

$$A = \begin{pmatrix} -1 & 1 & -2 \\ 1 & 1 & 1 \\ 0 & -2 & 1 \end{pmatrix}$$

and consider the augmented system

$$\begin{pmatrix} -1 & 1 & -2 & | & x \\ 1 & 1 & 1 & | & y \\ 0 & -2 & 1 & | & z \end{pmatrix}$$

where x, y, z are variables. This system is consistent precisely when (x, y, z) is in the span of u, v, w. We can put this matrix into echelon form via three moves. First exchange the first and second rows.

$$\begin{pmatrix} 1 & 1 & 1 & | & y \\ -1 & 1 & -2 & | & x \\ 0 & -2 & 1 & | & z \end{pmatrix}$$

Next, replace the second row with itself plus the first row.

$$\begin{pmatrix} 1 & 1 & 1 & | & y \\ 0 & 2 & -1 & | & x+y \\ 0 & -2 & 1 & | & z \end{pmatrix}$$

Finally, replace the third row with itself plus the second row

$$\begin{pmatrix} 1 & 1 & 1 & | & y \\ 0 & 2 & -1 & | & x+y \\ 0 & 0 & 0 & | & x+y+z \end{pmatrix}$$

Now that our augmented matrix is in echelon form, we can see that our system is consistent precisely when $x + y + z = 0$. Since the vector $(1, 1, 1)$ does not satisfy this property, it cannot be in the span of u, v and w. However, note that v is not a multiple of u, so the span cannot be just a line. Thus it must be a plane.

(b) We can see from our solution that the equation $x+y+z = 0$ precisely defines which vectors (x, y, z) are in the span. This is our equation.

(c) Consider the equation $x + y + z = 3$. What is its solution set? We know, by what we have seen in class, that its solution set is that of the homogeneous equation $x + y + z = 0$, plus any particular solution of $x + y + z = 3$. The solution set of $x + y + z = 0$ is the blue plane, and the point $(0, 0, 3)$ is a solution to $x + y + z = 3$. Thus the soution set of $x + y + z = 3$ is the blue plane plus the vector $(0, 0, 3)$, which is exactly the green plane. We deduce that the equation we are looking for is $x+y+z = 3$.

(5) (a) Explain in detail why the columns of a 10×13 matrix span $\mathbb{R}^{10}$ if and only if every row contains a pivot position.
(b) Explain why the columns of a 10×7 matrix cannot span $\mathbb{R}^{10}$.

Solution: Before we discuss either problem, let's prove the following fact. Suppose that A is an $m \times n$ matrix and B is row equivalent to A. Then the columns of A span $\mathbb{R}^m$ if and only if the same thing is true of B. To begin the proof, suppose that the columns of A span $\mathbb{R}^m$. We want to show that the columns of B span $\mathbb{R}^m$, which is the same thing as saying that $Bx = b$ has a solution for every b. Form the augmented system $[B \mid b]$. Applying row operations, we can turn this into $[A \mid c]$, where c is some new vector. Since the columns of A span $\mathbb{R}^m$, this system is consistent, and hence the same must have been true for the original system $[B \mid b]$.

(a) Since every matrix is row equivalent to its reduced echelon form, and by the observation above, it will suffice to assume that our matrix is in reduced echelon form. A reduced echelon form matrix has columns which are the vectors $e_1, e_2, e_3, \cdots, e_k$. These vectors span $\mathbb{R}^{10}$ if and only if k goes up to 10, which means that there cannot be any zero rows, and for a matrix in reduced echelon form, a zero row is exactly the same as a row without a pivot position.

(b) Let's again use the result we just proved. Put the matrix in reduced echelon form. The column vectors are $e_1, \cdots, e_k$, where k is at most 7, although it might be less if there are zero rows. These vectors certainly do not span $\mathbb{R}^{10}$, and so the column vectors of our original matrix cannot either.